Approximating the Growth Optimal Portfolio with a Diversified World Stock Index

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Abstract. This paper constructs and compares various total return world stock indices based on daily data. Due to diversification these indices are noticeably similar. A diversification theorem identifies any diversified portfolio as a proxy for the growth optimal portfolio. The paper constructs a diversified world stock index that outperforms a number of other indices and argues that it is a good proxy for the growth optimal portfolio. This has applications to derivative pricing and investment management.

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1 Introduction

In investment management there is a vital interest in identifying best performing portfolios. Theoretically, it can be shown that the *growth optimal portfolio* (GOP), which maximizes expected logarithmic utility from terminal wealth, is the portfolio that almost surely outperforms all other strictly positive portfolios after a sufficiently long time. This fascinating property of the GOP was discovered by Kelly (1956). It is also an optimal portfolio in a number of other senses as discussed in Platen & Heath (2006). The long term outperformance of any strictly positive portfolio by the GOP has been studied, for instance, in Latané (1959), Breiman (1961), Markowitz (1976) and Long (1990). In principle, the GOP is the portfolio that cannot be beaten in any reasonable systematic way. Reviews of this portfolio can be found in Hakansson & Ziemba (1995) and Platen (2005b).

Diversification is a classical concept in portfolio optimization, which has been applied in practice for centuries. Some results on different notions of diversification and diversified portfolios can be found, for instance, in Björk & Nåslund (1998), Hofmann & Platen (2000), Fernholz (2002), Platen (2004, 2005b) and Guan, Liu & Chong (2004). Under rather general assumptions it follows that diversified portfolios (DPs), where the fractions that are invested in different securities are small, behave similarly, see Platen & Heath (2006). If the market possesses a regularity property, under which the GOP is itself a DP, then in a large market the GOP can be asymptotically approximated by DPs. This result on diversification has been derived for continuous markets in Platen (2004) and for the case of jump diffusion markets in Platen (2005a). Since it does not assume any particular market dynamics, it endows the construction of a diversified proxy for the GOP with a robustness property.

The aim of this paper is to construct diversified world stock market indices from sector stock market index data. We shall argue that such diversified indices are reasonable proxies for the GOP and can be potentially used as enhanced index funds, see Scowcroft & Sefton (2003).

The paper is organized as follows: Section 2 introduces a jump diffusion financial market. Section 3 presents the GOP and points out that in the long run it is pathwise the best performing portfolio. Section 4 demonstrates that DPs approximate the GOP. Section 5 constructs various diversified world stock indices using daily sector stock index data and suggests a good proxy of the GOP. Section 6 estimates GOP fractions and discusses related issues. Section 7 compares indices and discusses their construction. Section 8 applies the constructed index to log-return estimation.
## 2 Financial Market Model

This section introduces a jump diffusion financial market along the lines of the benchmark approach as described in Platen (2002, 2006b) and Platen & Heath (2006). This approach presents a unified framework for financial modeling, investment management, derivative pricing and risk measurement.

We model a financial market that evolves on an infinite time horizon \( \mathbb{R}_+ = [0, \infty) \) and in which there are \( d \in \mathbb{N} = \{1, 2, \ldots\} \) sources of trading uncertainty. These are defined on a filtered probability space \((\Omega, \mathcal{A}, \mathcal{F}, P)\) where filtration \( \mathcal{F}_t = (\mathcal{F}_t)_{t \in \mathbb{R}_+} \) satisfies the usual conditions and where \( \mathcal{F}_0 \) is the trivial \( \sigma \)-algebra.

As usual, we regard \( \mathcal{F}_t \) as modeling the information available at time \( t \). Continuous trading uncertainties are represented by \( m \in \{1, 2, \ldots, d\} \) independent standard Wiener processes \( B^k_t = \{B^k_t, t \in \mathbb{R}_+\} \), for \( k \in \{1, 2, \ldots, m\} \). Event-driven trading uncertainties are modeled by \( d - m \) counting processes of the form \( p^k_t = \{p^k_t, t \in \mathbb{R}_+\} \), for \( k \in \{m+1, \ldots, d\} \), whose intensities \( h^k_t = \{h^k_t, t \in \mathbb{R}_+\} \) are predictable positive processes satisfying

\[
\int_0^t h^k_s \, ds < \infty \quad (2.1)
\]

almost surely, for all \( t \in \mathbb{R}_+ \). The corresponding compensated normalized jump martingales \( q^k_t = \{q^k_t, t \in \mathbb{R}_+\} \), for \( k \in \{m+1, \ldots, d\} \), are represented by the stochastic differentials

\[
dq^k_t = (h^k_t)^{-\frac{1}{2}} (dp^k_t - h^k_t \, dt) \quad (2.2)
\]

for \( t \in \mathbb{R}_+ \). Thus, the trading uncertainties are modeled by the vector process \( \mathbf{W} = \{W_t = (W^1_t, \ldots, W^m_t, W^{m+1}_t, \ldots, W^d_t), t \in \mathbb{R}_+\} \) of independent \((\mathcal{F}, P)\)-martingales with \( W^1_t = B^1_t, \ldots, W^m_t = B^m_t \) and \( W^{m+1}_t = q^{m+1}_t, \ldots, W^d_t = q^d_t \) for all \( t \in \mathbb{R}_+ \). Note that the conditional variance of the \( k \)th source of trading uncertainty is

\[
E \left( \left( W^k_t - W^k_s \right)^2 \bigg| \mathcal{F}_s \right) = s \quad (2.3)
\]

for all \( k \in \{1, 2, \ldots, d\} \) and \( s, t \in \mathbb{R}_+ \).

We now specify \( d + 1 \) primary security accounts, one of which is a locally riskless savings account \( S^0 = \{S^0_t, t \in \mathbb{R}_+\} \), given by

\[
S^0_t = \exp \left\{ \int_0^t r_s \, ds \right\} < \infty \quad (2.4)
\]

for \( t \in \mathbb{R}_+ \). Here \( r = \{r_t, t \in \mathbb{R}_+\} \) denotes the adapted short rate process. There are also \( d \) nonnegative risky primary security accounts, whose value processes are denoted by \( S^j = \{S^j_t, t \in \mathbb{R}_+\} \), for \( j \in \{1, 2, \ldots, d\} \). These are typically stocks, with all dividends reinvested. In Section 5 we shall choose them to be sector world stock indices. Note that foreign savings accounts, bonds, corporate bonds
and other derivatives may potentially also form primary security accounts. We assume that the dynamics of the risky assets satisfy the SDEs

\[
dS_t^j = S_t^j \left( a_t^j dt + \sum_{k=1}^d b_t^{j,k} dW_t^k \right),
\]

for \( t \in \mathbb{R}_+ \), \( S_0^j > 0 \), for all \( j \in \{1, 2, \ldots, d\} \). Here we denote the appreciation rate vector process by \( a = (a_t^1, \ldots, a_t^d)^	op, t \in \mathbb{R}_+ \) and the volatility matrix process by \( b = (b_t^{j,k})_{j,k=1}^d, t \in \mathbb{R}_+ \). We assume that \( b_t \) is an invertible matrix for every \( t \in \mathbb{R}_+ \), with inverse \( b_t^{-1} \). Furthermore, \( a_t^j \) and \( b_t^{j,k} \) are almost surely finite and predictable, ensuring that a unique strong solution of the system of SDEs (2.5) exists. The inequality

\[
b_t^{j,k} \geq -\sqrt{h_t^k}
\]

for all \( t \in \mathbb{R}_+ \) and \( k \in \{m+1, \ldots, d\} \), guarantees nonnegativity for each of the risky assets.

We can now define the market price of risk vector

\[
\theta_t = (\theta_t^1, \ldots, \theta_t^d)^	op = b_t^{-1}(a_t - r_t 1),
\]

for \( t \in \mathbb{R}_+ \). Here \( 1 = (1, \ldots, 1)^	op \) denotes the vector of ones. To ensure that the market is reasonable, in the sense that it is worth investing in risky assets, we assume that

\[
\theta_t^k < \sqrt{h_t^k}
\]

for all \( t \in \mathbb{R}_+ \) and \( k \in \{m+1, \ldots, d\} \). Furthermore, the total market price of risk is taken to be nonzero and finite, that is,

\[
|\theta_t| = \sqrt{\sum_{k=1}^d (\theta_t^k)^2} \in (0, \infty).
\]

Finally, we assume that

\[
\sum_{k=1}^d \sum_{j=0}^d \theta_t^k b_t^{-1,j,k} > 0
\]

for all \( t \in \mathbb{R}_+ \). Using the expression (2.7) for the market prices of risk, we can rewrite the SDE (2.5) for \( S_t^j \) in the form

\[
dS_t^j = S_t^j \left( r_t dt + \sum_{k=1}^d b_t^{j,k} (\theta_t^k dt + dW_t^k) \right) \quad (2.11)
\]

\[
= S_t^j \left( r_t dt + b_t^j (\theta_t dt + dW_t) \right) \quad (2.12)
\]

for \( t \in \mathbb{R}_+ \). Here we set \( b_t^j = (b_t^{j,1}, \ldots, b_t^{j,d}), \) for all \( t \in \mathbb{R}_+ \) and all \( j \in \{1, 2, \ldots, d\} \).
Let $S^\delta = \{S^\delta_t, t \in \mathbb{R}_+\}$ denote the value process of a portfolio, so that

$$S^\delta_t = \sum_{j=0}^{d} \delta^j_t S^j_t,$$  \hspace{1cm} (2.13)

for $t \in \mathbb{R}_+$. Here $\delta^j_t$ represents the number of units of the $j$th primary security account held in the portfolio at time $t$. We call the predictable stochastic process $\delta = \{\delta^0_t, \delta^1_t, \ldots, \delta^d_t\}, t \in \mathbb{R}_+$ a strategy if the Itô integral

$$\int_{0}^{t} \delta^j_s dS^j_s$$  \hspace{1cm} (2.14)

exists for each $j \in \{0, 1, \ldots, d\}$ and $t \in \mathbb{R}_+$. The portfolio $S^\delta$ is said to be self-financing if all changes in its value are due to gains or losses from trade in the primary security accounts. This can be expressed as follows:

$$dS^\delta_t = \sum_{j=0}^{d} \delta^j_t dS^j_t.$$  \hspace{1cm} (2.15)

In this paper we restrict our attention to self-financing portfolios.

Denote by $\mathcal{V}^+$ the set of strictly positive portfolio processes. For a given strategy $\delta$ with strictly positive portfolio process $S^\delta \in \mathcal{V}^+$, denote by $\pi_{\delta,t} = (\pi^1_{\delta,t}, \ldots, \pi^d_{\delta,t})^\top$ the vector of fractions of wealth invested in the primary security accounts. For a portfolio $S^\delta$, $\pi^j_{\delta,t}$ is the fraction of wealth held in the $j$th primary security account at time $t$:

$$\pi^j_{\delta,t} = \frac{\delta^j_t S^j_t}{S^\delta_t}$$  \hspace{1cm} (2.16)

for all $t \in \mathbb{R}_+$ and all $j \in \{0, 1, \ldots, d\}$. Note that these fractions can be negative, but they always sum to one, that is $\sum_{j=0}^{d} \pi^j_{\delta,t} = 1$. By equation (2.15) $S^\delta_t$ satisfies the following SDE, in terms of fractions:

$$dS^\delta_t = S^\delta_{t^-} \left( r_t dt + \sum_{j=0}^{d} \pi^j_{\delta,t} b_j (\theta_t dt + dW_t) \right).$$  \hspace{1cm} (2.17)

It follows by (2.6) that $S^\delta$ remains strictly positive if

$$\sum_{j=1}^{d} \pi^j_{\delta,t} b^t_j > -\sqrt{b^k_t}$$  \hspace{1cm} (2.18)

almost surely, for all $k \in \{m + 1, \ldots, d\}$ and $t \in \mathbb{R}_+$. We assume that all primary security accounts can jump to zero at any time. This is a realistic assumption, even though the intensities of such default events may be very small. This implies that all fractions of wealth in a strictly positive portfolio $S^\delta \in \mathcal{V}^+$ must be nonnegative, that is

$$\pi^j_{\delta,t} \geq 0$$  \hspace{1cm} (2.19)

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almost surely, for all \( j \in \{0, 1, \ldots, d\} \) and \( t \in \mathbb{R}_+ \).

Now, applying the Itô formula yields the following SDE for the logarithm of \( S_t^\delta \):

\[
\begin{align*}
    d\ln(S_t^\delta) &= g_t^\delta \, dt + \sum_{k=1}^d \sum_{j=1}^d \pi_{\delta,t}^{j,k} b_t^{j,k} \, dW_k^t + \sum_{k=m+1}^d \ln \left( 1 + \sum_{j=1}^d \pi_{\delta,t}^{j,k} \frac{b_t^{j,k}}{\sqrt{h_t^{k}}} \right) \, dW_k^t, \\
    &\quad \text{for } t \in \mathbb{R}_+.
\end{align*}
\]

(2.20)

where the growth rate \( g_t^\delta \) is expressed as

\[
\begin{align*}
g_t^\delta &= r_t + \sum_{k=1}^m \left( \sum_{j=1}^d \pi_{\delta,t}^{j,k} \theta_t^k - \frac{1}{2} \left( \sum_{j=1}^d \pi_{\delta,t}^{j,k} b_t^{j,k} \right)^2 \right) \\
    &\quad + \sum_{k=m+1}^d \left( \sum_{j=1}^d \pi_{\delta,t}^{j,k} \theta_t^k - \sqrt{h_t^k} \right) \left( \theta_t^k - \frac{1}{2} \left( \theta_t^k \right)^2 + \ln \left( 1 + \sum_{j=1}^d \pi_{\delta,t}^{j,k} \frac{b_t^{j,k}}{\sqrt{h_t^{k}}} \right) h_t^k \right),
\end{align*}
\]

(2.21)

for \( t \in \mathbb{R}_+ \).

## 3 Growth Optimal Portfolio

The growth optimal portfolio (GOP) is the central building block of the benchmark approach, see Platen & Heath (2006). It is the portfolio that maximizes growth over all strictly positive portfolios.

**Definition 3.1** A strictly positive portfolio process \( S_{t^*}^\delta \in \nu^+ \) is called a GOP if \( g_t^\delta \leq g_t^\delta^* \) almost surely, for all \( t \in \mathbb{R}_+ \) and \( S_t^\delta \in \nu^+ \).

By equation (2.17), see Platen (2006a) and Platen & Heath (2006), it follows that the GOP satisfies the SDE

\[
\begin{align*}
    dS_{t^*}^\delta &= S_{t^*}^\delta \left( r_t \, dt + \sum_{k=1}^m \theta_t^k (\theta_t^k \, dt + dW_t^k) + \sum_{k=m+1}^d \frac{\theta_t^k}{1 - \theta_t^k (h_t^k)^{-1/2}} (\theta_t^k \, dt + dW_t^k) \right),
\end{align*}
\]

(3.1)

for \( t \in \mathbb{R}_+ \), with \( S_0^\delta > 0 \). Since a GOP is uniquely determined by (3.1), up to its initial value, which we set to \( S_0^\delta^* = 100 \), we call \( S_{t^*}^\delta \) the GOP. The optimal growth rate of the GOP is given by

\[
\begin{align*}
g_{t^*}^\delta &= r_t + \frac{1}{2} \sum_{k=1}^m (\theta_t^k)^2 - \sum_{k=m+1}^d h_t^k \left( \ln \left( 1 + \frac{\theta_t^k}{\sqrt{h_t^k}} - \theta_t^k \right) + \frac{\theta_t^k}{\sqrt{h_t^k}} \right),
\end{align*}
\]

(3.2)

for all \( t \in \mathbb{R}_+ \). The corresponding vector of optimal fractions is

\[
\pi_{\delta,t} = (\pi_{\delta,t}^1, \ldots, \pi_{\delta,t}^d)^\top = (b_t^{-1})^\top c_t, \tag{3.3}
\]
where the predictable vector process \( c = \{ c_t = (c^1_t, \ldots, c^d_t)^\top, t \in \mathbb{R}_+ \} \) is given by
\[
c^k_t = \begin{cases} 
\theta^k_t & \text{for } k \in \{1, 2, \ldots, m\} \\
\frac{\theta^k_t}{1-\rho^k_t(h^k_t)-1/2} & \text{for } k \in \{m+1, \ldots, d\} 
\end{cases}
\] (3.4)
for \( t \in \mathbb{R}_+ \). Since the GOP is a strictly positive portfolio, by (2.19) we know that the optimal fractions of the GOP are nonnegative:
\[
\pi^j_{\delta, t} \geq 0 \quad (3.5)
\]
for all \( t \in \mathbb{R}_+ \) and all \( j \in \{0, 1, \ldots, d\} \).

It is shown in Platen & Heath (2006) that the GOP has remarkable properties which single it out as the best performing portfolio according to various objectives. For instance, it has the maximum long term growth rate. That is, after a sufficiently long time the trajectories of the GOP almost surely outperform those of every other portfolio. This fascinating property is summarized in the following statement, see Platen & Heath (2006) for the proof:

**Theorem 3.2** The GOP \( S^\delta \) has the largest long term growth rate among all strictly positive portfolios \( S^\delta \in \mathcal{U}^+ \) in the sense that
\[
\limsup_{T \to \infty} \frac{1}{T} \ln \left( \frac{S^\delta_T}{S^\delta_0} \right) \leq \limsup_{T \to \infty} \frac{1}{T} \ln \left( \frac{S^\delta_T}{S^\delta_{\ast}} \right) \quad (3.6)
\]
almost surely.

Due to the result of Theorem 3.2 any investor who has a sufficiently long time horizon should invest according the GOP. Theoretically, if one has calibrated an appropriate model, then one can determine the optimal fractions according to (3.3). However, as pointed out in Frankfurter, Phillips & Seagle (1971), Merton (1980), Jorion (1986) and Broadie (1993) it is unrealistic to assume that one can estimate reliably expected returns from available data.

### 4 Diversified Portfolios and Approximate GOPs

Since the sufficiently accurate estimation of the fractions of the GOP seems in practice not to be feasible we will describe a result that allows its approximation. More precisely, we describe a diversification theorem that will allow us to identify proxies for the GOP. For each \( d \in \mathcal{N} \), let \( \mathcal{S}_{(d)} \) denote the market comprising \( S^0_t \) and the \( d \) risky assets \( S^1_t, \ldots, S^d_t \). In the sequel, \( S^d_{(d)}(t) \) denotes the value of a self-financing portfolio made up of assets from \( \mathcal{S}_{(d)} \).
Definition 4.1 A sequence of portfolios \((S^d)_{d \in \mathcal{N}}\) is said to be a sequence of diversified portfolios (DPs), if there exist strictly positive constants \(K_1, K_2\) and \(K_3 \in (0, \infty)\), independent of \(d\), such that, for all \(d \in \{K_3, K_3 + 1, \ldots\}\) one has
\[
|\pi^j_{d,t}| \leq \frac{K_2}{d^{1/2} + K_1} \tag{4.1}
\]
almost surely, for all \(j \in \{0, 1, \ldots, d\}\) and all \(t \in \mathbb{R}_+\).

Intuitively, this means that in large markets a DP invests only small fractions in each primary security account. Alternative definitions of diversification can be found in Litterman and the Quantitative Researches Group (2003), Luenberger (1997) or Fernholz (2002). We now define the so-called specific volatilities by setting
\[
\sigma_{j,k}^{j,k}(t) = \begin{cases} 
\theta_k^j - b_k^{j,k} & \text{if } k \in \{1, 2, \ldots, m\} \\
\theta_k^j - b_k^{j,k} \left(1 - \frac{\theta_k^j}{\sqrt{h_k^j}}\right) & \text{if } k \in \{m + 1, \ldots, d\} 
\end{cases} \tag{4.2}
\]
for all \(t \in \mathbb{R}_+\) and all \(j, k \in \{1, 2, \ldots, d\}\). We also set \(\sigma_{0,k}^{0,k}(t) = \theta_k^j\) for all \(t \in \mathbb{R}_+\) and \(k \in \{1, 2, \ldots, d\}\). The specific volatilities are required to be finite. That is, for all \(d \in \mathcal{N}, T \in \mathbb{R}_+\) and \(j \in \{0, 1, \ldots, d\}\), we suppose that
\[
\int_0^T \sum_{k=1}^d \sigma_{j,k}^{j,k}(t)^2 \, dt \leq C_T < \infty \tag{4.3}
\]
almost surely, where \(C_T\) is some finite \(\mathcal{A}_T\)-measurable random variable, independent of \(d\). Furthermore, it is assumed that
\[
\sigma_{j,k}^{j,k}(t) < \sqrt{h_k^j} \tag{4.4}
\]
holds almost surely, for all \(t \in \mathbb{R}_+, k \in \{m + 1, \ldots, d\}\) and \(j \in \{0, 1, \ldots, d\}\). The total specific volatility with respect to the \(k\)th source of trading uncertainty is at time \(t\) defined by
\[
\hat{\sigma}_k^d(t) = \sum_{j=0}^d |\sigma_{j,k}^{j,k}(t)| \tag{4.5}
\]
for all \(t \in \mathbb{R}_+\) and all \(k \in \{1, 2, \ldots, d\}\).

To establish the diversification theorem we require the following regularity property:

Definition 4.2 A sequence of markets \((S^d)_{d \in \mathcal{N}}\) is called regular if there exists a constant \(K_5 \in (0, \infty)\), independent of \(d\), such that
\[
E\left(\left(\hat{\sigma}_k^d(t)\right)^2\right) \leq K_5 \tag{4.6}
\]
for all \(t \in \mathbb{R}_+, d \in \mathcal{N}\) and \(k \in \{1, 2, \ldots, d\}\).
The property above can be interpreted as saying that each source of trading uncertainty influences only a restricted number of primary security accounts, when these are denominated in units of the GOP. From now on we assume that \((S_{(d)})_{d \in \mathcal{N}}\) is a regular sequence of markets.

For given \(d \in \mathcal{N}\) and a portfolio \(S_{(d)}^\delta\) in the market \(S_{(d)}\), the tracking rate \(R_{(d)}^\delta(t)\) measures the distance from this portfolio to the GOP as follows:

\[
R_{(d)}^\delta(t) = \sum_{k=1}^{d} \left( \sum_{j=0}^{d} \pi_{j,k}^\delta \sigma_j(t) \right)^2
\]

(4.7)

for \(t \in \mathbb{R}_+\). One can show by (2.17), (3.1) and (4.7) that \(R_{(d)}^\delta(t) = 0\).

**Definition 4.3** A sequence \((S_{(d)}^\delta)_{d \in \mathcal{N}}\) of strictly positive portfolios is called a sequence of approximate GOPs if the corresponding sequence of tracking rates vanishes in probability. That is, for each \(\varepsilon > 0\) we have

\[
\lim_{d \to \infty} R_{(d)}^\delta(t) = 0 \quad (4.8)
\]

for all \(t \in \mathbb{R}_+\).

The following diversification theorem is proved in Platen (2005a).

**Theorem 4.4** For a regular sequence of markets \((S_{(d)})_{d \in \mathcal{N}}\), each sequence \((S_{(d)}^\delta)_{d \in \mathcal{N}}\) of DPs is a sequence of approximate GOPs.

Subject to the above regularity condition, this result states that any portfolio with small fractions in all the primary security accounts is a reasonable proxy for the GOP, if the market is large enough. It is highly significant that the validity of this statement is model independent.

5 Construction of Indices by Portfolio Generating Functions

Market indices are usually conceived as measures of general market performance. We aim to construct an index that measures the performance of the global world stock market. Usually such an index would be constructed by using the ratios of the market capitalization of each security in the market to the total market capitalization as the fractions for investment, see Scowcroft & Sefton (2003). Stocks with larger market capitalizations would then have larger fractions. This type of index is called a market capitalization weighted index (MCI) and is widely
used as a benchmark in investment management. However, since such an index tends to be dominated by a relatively small number of stocks, it is, in general, not a DP. Therefore, an MCI is unlikely to be an ideal proxy for the GOP.

Our primary objective is to construct an investable, diversified global stock market index, which eventually could be useful as an enhanced index fund. This index shall perform well in the long term and will be called world stock index (WSI). The construction of such a WSI shall be rule based and data driven. It should avoid subjective decisions.

In practice, the number of risky assets used to construct a portfolio is obviously finite and often not even all that large. In this paper we shall construct proxies for the GOP comprising 104 sector stock market accumulation indices and risky primary security accounts.

Since the fractions in the GOP are nonnegative, we only consider portfolios without short sales, that is, all fractions must be in $[0, 1]$. A very simple DP with nonnegative fractions is the equally weighted index (EWI) with

$$
\pi_{\text{EWI},t}^j = \frac{1}{d}
$$

for all $j \in \{1, 2, \ldots, d\}$.

Let $S^\delta = \{S^\delta_t, t \in \mathbb{R}_+\}$ be the value process of a given portfolio and denote by $\pi^\delta_t = (\pi^\delta_1, \ldots, \pi^\delta_d)^\top \in \mathbb{R}^d$ the vector of its fractions at time $t$. This portfolio could be, for example, the MCI or the portfolio corresponding to an asset allocation strategy that permits short sales. Now, define a portfolio generating function (PGF) $A : \mathbb{R}^d \to [0, 1]^d$ that maps the vector of fractions, $\pi^\delta_t \in \mathbb{R}^d$, into a vector of nonnegative fractions

$$
\pi^\delta_t = (\pi^1_{\delta,t}, \ldots, \pi^d_{\delta,t})^\top = A(\pi^\delta_t) \in [0, 1]^d
$$

such that $\sum_{j=1}^d \pi^j_{\delta,t} = 1$ for all $t \in \mathbb{R}_+$, see Fernholz (2002). Note that there is considerable freedom for choosing a PGF. Many different classes of portfolios can be generated in this way. In particular, we are interested in PGFs that make $S^\delta$ a DP in the sense of Definition 4.1. By virtue of Theorem 4.4 this leads to a sequence of approximate GOPs.

We now describe a useful example of a PGF. Let $p \in [0, 1]$ be an arbitrary constant. The $j$th fraction of (5.2) is given by

$$
\pi^j_{\delta,t} = \frac{(\pi^j_{\delta,t} + \mu_t)^p}{\sum_{l=1}^d (\pi^l_{\delta,t} + \mu_t)^p}
$$

for all $j \in \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}_+$, where

$$
\mu_t \geq \inf_{j} \pi^j_{\delta,t}.
$$
Note that the fractions of $S^\delta$ obtained in this way are nonnegative. If $\mu_t$ is sufficiently large, then $S^\delta$ will be a DP. It is also noted that as the portfolio size $d$ gets larger, the fractions of $S^\delta$ tend to $1/d$ which equals the fractions of the EWI, that is
\[
\lim_{d \to \infty} \pi_{j, t}^{\hat{\delta}} = \frac{1}{d} = \pi_{j, \text{EWI}, t}^{\hat{\delta}}.
\]
for all $t \in \mathbb{R}_+$.

The PGF in the above example has some similarity with a PGF introduced in Fernholz (2002), where the fractions of the MCI are used to construct a PGF as follows:
\[
\pi_{j, t}^{\hat{\delta}} = \frac{(\pi_{j, \text{MCI}, t}^{\delta})^p}{\sum_{l=1}^d (\pi_{l, \text{MCI}, t}^{\delta})^p}.
\]
for $j \in \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}_+$, where $p \in [0, 1]$. The resulting portfolio is called the diversity weighted index (DWI). For a given $p \in (0, 1)$, this PGF magnifies the small fractions of the MCI and reduces its large fractions. Therefore, it transforms the market portfolio into a more diversified portfolio. Still, (5.6) does not necessarily generate a DP in the sense of Definition 4.1. Note that in (5.6), if $p = 0$ the corresponding portfolio is the EWI and if $p = 1$ it is the MCI.

All PGFs that induce DPs with the same ranking of fractions generate portfolios with rather similar behavior. For instance, a PGF of exponential form, given by
\[
\pi_{j, t}^{\hat{\delta}} = \frac{\exp\{\lambda \pi_{j, t}^{\delta}\}}{\sum_{l=1}^d \exp\{\lambda \pi_{l, t}^{\delta}\}}.
\]
for all $j \in \{1, 2, \ldots, d\}$, $t \in \mathbb{R}_+$ and a value of $\lambda$ slightly greater than zero, produces almost the same dynamics as (5.3) with $p = 1$. In this case (5.7) is approximated by
\[
\pi_{j, t}^{\hat{\delta}} \simeq \frac{1 + \lambda \pi_{j, t}^{\delta}}{\sum_{l=1}^d (1 + \lambda \pi_{l, t}^{\delta})}.
\]
We emphasize that diversification yields a certain robustness. We shall demonstrate that small changes in the fractions in the direction of those of the GOP improve the long term performance of the resulting index.

6 Estimating GOP Fractions

The challenge is to identify potential changes in the estimated fractions of the GOP that may point in the direction of the true fractions of the GOP. Following Markowitz (1952, 1959), mean-variance portfolio optimization in our current continuous time jump diffusion market was performed in Platen (2006b). It turns out that the resulting efficient portfolios are always a combination of the savings
account and a mutual fund (MF) $S^{\delta_{MF}}$ which, in general, is different from the GOP. Its value satisfies the SDE

$$dS^{\delta_{MF}}_t = S^{\delta_{MF}}_t \left( r_t + \sum_{k=1}^d \theta^k_t (\theta^k_t dt + dW^k_t) \right)$$

(6.1)

for all $t \in \mathbb{R}_+$. Only when the ratios $\theta^k_t / \sqrt{h^k_t}$ vanish for all $k \in \{m+1, \ldots, d\}$ and $t \in \mathbb{R}_+$, does the MF coincide with the GOP. For simplicity, we assume that this is the case for the real stock market, with sector stock market indices taking the role of primary security accounts. It then follows from (3.3) and (3.4) that the vector of the fractions of the GOP satisfies

$$\pi^{\delta_{MF},t} = \pi^{\delta_{MF},t}_t = (b_t^{-1})^\top \theta_t = \Sigma_t^{-1} (a_t - r_t 1)$$

(6.2)

for $t \in \mathbb{R}_+$. In this formula $\Sigma_t = b_t b_t^\top$ is the covariance matrix of log-returns and $a_t$ is the vector of expected returns of the primary security accounts. Using the standard estimation method the covariance matrix can be estimated from one year’s worth of observed daily data, assuming that it does not change too much over the year. On each day we estimate in this manner the covariance matrix using the most recent one year observation period. Any estimate of the vector of expected returns is very unreliable however. Even under the simple Black-Scholes model it is impossible to obtain reasonable estimates for the expected returns based on the available data, see, for instance, Merton (1980). This is a key problem in the application of modern portfolio theory. Despite the fact that expected return estimates are unreliable, they may still contain some valuable information. Therefore, we will exploit potentially relevant information to date about expected returns by estimating their values from last year’s data, using the standard estimation technique.

Relying directly on estimated daily changes of expected returns would cause investors in the estimated GOP continually to make substantial adjustments to their portions. Also these adjustments are, in general, inexplicable and result in prohibitive transaction costs. Moreover, the GOP fractions estimated in such a manner can be substantially negative or positive. A portfolio constructed in this way, with such extreme fluctuations of its fractions, would not be pursued by any reasonable investment manager, see Scowcroft & Sefton (2003). With this in mind, we propose a different strategy for constructing a well-performing diversified WSI.

We use the fact that the GOP should be a DP with nonnegative fractions. To approximate it we first estimate the daily covariance matrix of log-returns and the daily vector of expected returns from the most recent one year of data. By (6.2) this yields an estimate for the vector of fractions of the GOP. By applying the PGF (5.3) with a sufficiently large level $\mu_t \in \mathbb{R}_+$ and an appropriate exponent $p \in [0, 1]$, we obtain fractions of a DP. Importantly, these fractions are nonnegative and have the same ranking as those estimated for the GOP.
The diversified index constructed, as described above, is then called the WSI. It is rather similar to the EWI. Indeed, as we shall see, the fluctuations of both the WSI and the EWI are almost identical for the chosen parameters in our example. However, differences arise in their long term performance, as will be shown below.

7 Comparison of Indices

By using the methodology described above we now construct the WSI whose fractions are determined by the PGF (5.3), with \( p = 1 \) and \( \mu_t = 1000 \). The constituents are 104 sector stock market total return indices, denominated in US dollars, as provided by DataStream Advance and their abbreviations are given at the end of the paper. The daily data used cover the period from 01 January 1973 to 31 August 2006. The data for the sector indices are displayed in Figure 1. The fractions of the WSI are comparable in magnitude. To illustrate this, we plot these in Figure 2. For comparison, we have also constructed the equally weighted index (EWI), the diversity weighted index (DWI) and the market capitalization weighted index (MCI) using Formula (5.6) with \( p = 0, \ p = 0.5 \) and \( p = 1 \), respectively. For convenience, all indices have the same initial value of 100 US dollars at the starting date. They are shown separately in Figure 3. The WSI does not put an emphasis on the level of market capitalization, but consistently aims to deduce the fractions of the GOP. In the long run it appears to outperform the DWI, the MCI and almost all sector indices. In our view this is not coincidental, but due to its proximity to the GOP. As shown in (5.5) for a large number of
constituents \( d \), the fractions of the WSI are approximately the same as those of the EWI. This results in that the EWI is also in the proximity of the GOP.

We note from Figure 3 that the DWI performed better in the long term than the MCI, but its performance was not as good as the EWI. This is not surprising since the PGF (5.6) places the fractions of the DWI between those of the MCI and the EWI.
We observe the performance of the WSI is slightly better than the EWI, see Figure 3, this is due to the small fluctuations in the fractions of the WSI and do not change drastically over time and are quite homogenous, see Figure 2. On the other hand, strict equal weighting of sector indices performed equally well as the WSI. This suggests that for a DP with large number of constituents small fluctuations in the fractions do not have a major impact on the long term performance of an index. However, the performance of the WSI can only be better but not worse than that of the EWI, which is due to the ranking of the estimated fractions of the WSI. It seems that the estimated weights of the GOP contain some information about the ranking of the true fractions of the GOP.

It appears that the better performance of the WSI is partly due to the diversified inclusion of the stock market indices of emerging sectors, in particular softwares and internet. Furthermore, the estimated weights of the GOP seem to contain some information about the ranking of the true fractions of the GOP.

8 Application of Constructed Index

As described in Platen & Heath (2006), there are many applications where the WSI can be used as proxy for the GOP. In particular, when the WSI dynamics is properly modeled it allows the real world pricing of contingent claims in an incomplete market. In that case the WSI is taken as numeraire and the pricing measure is chosen to be the real world probability measure. Derivative prices denominated in the WSI can be interpreted as real world martingales and the existence of an equivalent risk neutral probability measure is unnecessary. The WSI has many other applications. As indicated previously, it is a good candidate for an enhanced index fund with a short selling constraint.

Platen (2002) introduced, the so-called minimal market model, for the GOP and by extension, for the WSI. According to this model the daily log-returns of the GOP should be estimated to be Student $t$ distributed with approximately four degrees of freedom. This prediction was investigated empirically in Fergusson & Platen (2006). We apply the same methodology to the indices considered here. We find, as in Fergusson & Platen (2006), that in the wide class of symmetric generalized hyperbolic distributions the Student $t$ distribution offers the best fit for daily log-returns for all the above indices when compared with the normal-inverse Gaussian, hyperbolic and variance gamma distributions.

We plot the empirical Student $t$ log-return densities in log-scale in Figure 4. The estimated Student $t$ densities for the EWI, DWI, MCI and WSI have degrees of freedom 4.65, 4.62, 4.50, 4.65, respectively. Note that the degrees of freedom of the Student $t$ densities are close to the value predicted by the stylized minimal market model, see Platen & Heath (2006). This is a rewarding result because we already singled out the WSI as a good proxy for the GOP, due to its best long
term performance.

**Conclusion**

The proposed rule based method of constructing a proxy for the growth optimal portfolio has specific advantages over the methodologies of diversity weighting and market capitalization weighting. First, it relies entirely on observable and estimable information. Second, the approach is theoretically justified by searching for the diversified outperformance of the long-term growth rate. Lastly, it allows the investors to understand the index movements on the basis of expectations and covariances of log-returns of the underlying assets in a classical sense and also presents a consistent platform for constructing diversified world stock indices. The proposed methodology is very tractable and easy to implement.

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The WSI as shown in Figure 3 can be found at the webpage of the second author.
Abbreviation

Sector stock indices are abbreviated by DataStream Advance: BMATS, GASDS, INSUP, CHMSP, COMPH, FMFSH, HOMES, ELEQP, FORST, HVYCN, MEDAG, CNFIN, IMACH, DEFEN, HCPRO, FINAD, WAITE, OILEP, OILSV, PIPEL, TYRES, NOFMS, RECSV, STEEL, ELETR, DURHP, FORMS, TOYSG, NDRHP, AUPRT, TRNSV, AUTOS, APRET, BREWS, DISTV, CLTHG, COMPK, FDPRD, RISKS, CNELE, INYNK, PLTNM, TOBAC, HOTEL, RAILS, PAPER, PURLS, HIMP, BUSUP, BDRET, FDRST, SPRET, MTUHL, CHEMS, ALUMN, TRAVL, PRRMC, OILIN, AEROS, MARIN, GAMING, DIVIN, BANKS, MIKSP, ASSET, OFFEQ, LFINS, PCINS, INSBR, ITINT, NVSY, RLDEV, SPFTD, BRENQ, COMMV, SOFTD, BODEN, COMMV, GOLDS, HPRSL, COALM, DGRET, MINES, TELEG, AIRLN, SEMIC, TRUCK, MEDEQ, REINS, BUSTE, ELECT, FFLINS, TELMB, WATER, CMPSV, MORTF, REITS, RPPCR, SPCSV, FOOTW, DELSV, BIOTC, SOFTW, DIAMD, INTNT.

References


