On Financial Markets where only Buy-And-Hold Trading is possible

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This work is dedicated to the memory of our colleague and dear friend Nicola Bruti Liberati, who died tragically on the 28th of August, 2007.

Abstract. A financial market model where agents can only trade using realistic buy-and-hold strategies is considered. Minimal assumptions are made on the nature of the asset-price process — in particular, the semimartingale property is not assumed. Via a natural assumption of limited opportunities for unlimited resulting wealth from trading, coined the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition, we establish that asset-prices have to be semimartingales, as well as a weakened version of the Fundamental Theorem of Asset Pricing that involves supermartingale deflators rather than Equivalent Martingale Measures. Further, the utility maximization problem is considered and it is shown that using only buy-and-hold strategies, optimal utilities and wealth processes resulting from continuous trading can be approximated arbitrarily well.

0. Introduction

0.1. Background and significance. In the process for obtaining a sufficiently general version of the Fundamental Theorem of Asset Pricing (FTAP), semimartingales proved crucial in modeling asset-price processes. The powerful tool of stochastic integration with respect to general predictable integrands, that semimartingales are exactly tailored for, finally lead to the culmination of the theory in [DS94, DS98]. The FTAP connects the economically sound notion of No Free Lunch with Vanishing Risk (NFLVR) with the mathematical concept of existence of an Equivalent Martingale Measure (EMM), i.e., an auxiliary probability, equivalent to the original (in the sense that they have the same impossibility events), that makes the asset-price processes have some kind of martingale property. This viability of the financial market that is ensured by the NFLVR property leads in turn to a very satisfactory solution to the utility maximization problem from terminal wealth in a general modeling environment as is described in full in [KS99, KS03].

For the above approach to work one has to utilize stochastic integration using general predictable integrands, which translates to allowing for continuous-time trading in the financial market. Continuous-time trading is of vast theoretical importance, since it allows...
for elegant representations of optimal hedging and trading strategies. In practice, however, it is only an ideal approximation; the only feasible way of trading is via simple \textit{buy-and-hold} strategies. Therefore, it is natural to question the usefulness of such modeling approach, especially in the context of numerical approximations, where discretization is inevitable.

Furthermore, it has recently been argued that existence of an EMM is \textit{not} necessary for viability of the market; to this effect, see [LW00, Pla02, FKK05]. Even in cases where classical arbitrage opportunities are present in the market, credit constraints will not allow for arbitrages to be scaled to any desired degree. (More surprisingly, it is possible for a utility-maximizing economic agent to consider an arbitrage \textit{suboptimal} as an investment strategy — see §4.3.3 of [KK07] for an example). It is rather the existence of a \textit{supermartingale deflator} (see Definition 2.2), a concept weaker than existence of an EMM, that allows for a consistent theory to be developed.

Our purpose in this work is to provide answers to the following questions:

\begin{enumerate}
\item Why is the use of semimartingales to model asset-price processes crucial?
\item Is there an analogous result to the FTAP that involves weaker, both economic and mathematical conditions and does not require the heavy use of general stochastic integration, but only assumes the possibility of buy-and-hold trading?
\item Are the optimal-wealth results obtained by allowing continuous trading useful? That is, can they be sufficiently approximated via \textit{buy-and-hold} trading?
\end{enumerate}

A partial, but rather precise, answer to question (1) is already present in [DS94]; here, we give a more general answer under weaker assumptions. A thorough comparison is carried out in §2.4.3. A different approach, obtaining the semimartingale property of the asset-price processes using finite value for the expected utility maximization problem, is undertaken in [AI05]. However, conditions involving finite expected utility are only \textit{sufficient} to ensure the asset prices are semimartingales; here, we discuss conditions that are both necessary and sufficient. The weakened version of the FTAP that we shall come up with as an answer to question (2) is a “buy-and-hold, no-short-sale trading” version of Theorem 4.12 from [KK07]. We also provide a positive answer to question (3), opening the way to the use of approximate optimization methods.

**0.2. Organization and results.** Section 1 introduces the market model, buy-and-hold trading and no-short-sale constraints. Section 2 begins by introducing the condition of \textit{No Unbounded Profit with Bounded Risk} (NUPBR), a weakening of the NFLVR condition, as well as the concept of \textit{supermartingale deflators}. After this, Theorem 2.3, our first main result, is formulated. This states the equivalence of the semimartingale property of the asset-price processes, existence of a supermartingale deflator and the NUPBR condition when only buy-and-hold no-short-sale strategies are involved. Moving further, Theorem 3.1 of Section 3 deals with an equivalent of Theorem 2.3 in the case of continuous asset-price processes where complete trading freedom is allowed in the class of buy-and-hold strategies. There is an important difference from the constrained case which, we feel, gives more value to Theorem 2.3 from a practical point of view. In Section 4 we visit
the utility maximization problem and show in Theorem 4.1 that, under weak economic assumptions, optimal strategies using buy-and-hold trading approximate arbitrarily well their continuous-trading counterparts. Sections 5 and 6 deal with proving the aforementioned results. In fact, Section 5 contains an interesting result on “multiplicative” approximation of positive stochastic integrals, following in effect the proportional, rather than the absolute, continuous trading strategy. We note that, though hidden in the background, the proofs of our results depend heavily on the notion of the numéraire portfolio (also called growth-optimal, log-optimal or benchmark), as it appears in a series of works; [Kel56, Lon90, Bec01, GK03, Pla02, PH06, Pla06, KK07, CL07], to mention a few.

1. The Financial Market and Trading

1.1. The financial market model. In all that follows, the random movement of $d$ risky assets in the market is modeled via arbitrary stochastic processes $S^1, \ldots, S^d$. There also exists another special asset, whose price-process is strictly positive and denoted by $S^0$; this asset is considered a “baseline”, in that all other assets are denominated in units of $S^0$. As is usual in the field of Mathematical Finance, we assume that assets have already been discounted by $S^0$, i.e., $S^0 \equiv 1$. The above processes are defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a filtration satisfying $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \in \mathbb{R}$, as well as the usual assumptions of right-continuity and saturation by all $\mathbb{P}$-null sets of $\mathcal{F}$.

There is no a priori assumption about the asset-price process $S := (S^1, \ldots, S^d)$ being a semimartingale. This property will come as a consequence of some natural assumption that will be introduced later. The following minimal restriction on $S$ will be in force throughout.

**Assumption 1.1.** For $i = 1, \ldots, d$, the stochastic process $S^i$ is nonnegative, $\mathbb{F}$-adapted, càdlàg (right-continuous with left-hand limits) and remains at zero if it reaches zero.

The above assumption is very natural on economic grounds. Usually, for each $i = 1, \ldots, d$, $S^i$ denotes the discounted cum-dividend share price process of some company with limited liability, which ensures its nonnegativity. If some company goes bankrupt, then it stops functioning and its future value remains zero.

**Remark 1.2.** In mathematically precise terms, the last requirement in Assumption 1.1 is formalized as follows. For any $i = 1, \ldots, d$, define $\zeta^i := \inf\{t \in \mathbb{R}_+ : S^i_t = 0\}$ to be the lifetime of the $i$th asset. We then ask that $S^i_t = 0$ for all $t \in [\zeta^i, \infty)$ on $\{\zeta^i < \infty\}$.

1.2. Trading via buy-and-hold strategies. In the market described above, economic agents can trade in order to reallocate their wealth. We shall be denoting generically by $T$ a collection of trading times of the form $\{0 = \tau_0 < \tau_1 < \ldots < \tau_n = T\}$, where each $\tau_j$, $j = 0, \ldots, n$, is a finite $\mathbb{F}$-stopping time and the typically random $n$ ranges in the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. The physical interpretation of times in $T$ is that these are instances when some given economic agent may trade in the market. Below we shall soon elaborate further on this point. The random time $T = \sup T$ is the (agent-specific)
financial planning horizon, by which we shall always mean some finite stopping time. For a given financial planning horizon $T$, the class of all possible collections $T$ with $T = \sup T$ shall be denoted by $\mathcal{F}$. We then let $\mathcal{F}$ denote the union of all the classes $\mathcal{F}^T$ when $T$ ranges through all finite stopping times. For each $\{\tau_0 < \tau_1 < \ldots < \tau_n\} = T \in \mathcal{F}$, it is assumed that there exists an agent in the market that may trade at the discrete instances $\tau_0, \tau_1, \ldots, \tau_{n-1}$, while $\tau_n$ will be the time of wealth assessment (the agent will stop trading and collect whatever wealth has been obtained up to that point). We shall call this form of trading buy-and-hold, in contrast with continuous trading where one is able to change the position in the assets in a continuous fashion. This last form of trading is only of theoretical value, since it cannot be implemented in reality, even if one ignores market frictions, as we do here to keep the exposition simple.

We now describe in more detail how trading takes place. Fix $T = \{\tau_0 < \tau_1 < \ldots < \tau_n\}$ and consider some economic agent who may invest only at the times included in $T$. This specific agent will decide at each instant $\tau_{j-1}$, $j = 1, \ldots, n$, to hold a number $\vartheta^{i,j}_{\tau_{j-1}}$ from asset $i$ until the next potential trading time. Call $\vartheta_{\tau_{j-1}} := (\vartheta^{i,j}_{\tau_{j-1}})_{i=1,\ldots,d}$; it is assumed throughout that $\vartheta_{\tau_{j-1}}$ is $\mathcal{F}_{\tau_{j-1}}$-measurable in order to model absence of clairvoyance and insider trading. Starting from initial capital $x \in \mathbb{R}_+$ and following the strategy described by the predictable process $\vartheta := \sum_{j=1}^n \vartheta^{i,j}_{\tau_{j-1}}[\tau_{j-1},\tau_j]$, the agent’s wealth at time $t \in \mathbb{R}$ is

\begin{equation}
X^{x,\vartheta}_t := x + \sum_{j=1}^n \langle \vartheta^{j,1}_{\tau_{j-1}}, S_{\tau_{j-1}} - S_{\tau_{j-1} \wedge \tau} \rangle = x + \sum_{j=1}^n \sum_{i=1}^d \vartheta^{i,j}_{\tau_{j-1}} (S^i_{\tau_{j-1} \wedge \tau} - S^i_{\tau_{j-1} \wedge \tau}).
\end{equation}

Observe that $X^{x,\vartheta}_t = X^{x,\vartheta}_{\tau_n}$ for all $t \in [\tau_n, \infty)$, which agrees with our interpretation of time $\tau_n$ as the time that trading stops for the agent trading at times included in $T$.

In view of Assumption 1.1, each wealth process $X^{x,\vartheta}$, as defined in (1.1), is càdlàg and adapted, but could in principle become negative. This has to be disallowed based on economic reasoning, since it corresponds to bankruptcy of the agent who should, therefore, be refrained from investing further. We then call a wealth process $X$ admissible if it satisfies\(^1\) $X \geq 0$.

For each $\mathcal{T} \in \mathcal{F}$ and $x \in \mathbb{R}_+$, let $\mathcal{X}(x; \mathcal{T})$ denote the set of all admissible wealth processes that start from initial capital $x$ and trade at times in $\mathcal{T}$. Set also $\mathcal{X}(x, T) := \bigcup_{\mathcal{T} \in \mathcal{F}} \mathcal{X}(x; \mathcal{T})$ to be the set of all possible wealth processes that can be achieved starting from $x$, having financial planning horizon equal to $T$ and using some buy-and-hold strategy. Finally, let $\mathcal{X}$ denote the set of all possible wealth processes, starting from any capital $x \in \mathbb{R}_+$ and having any financial planning horizon. Observe that $\mathcal{X}(x; \mathcal{T}) = x\mathcal{X}(1; \mathcal{T})$ for all $x \in \mathbb{R}_+$ and $\mathcal{T} \in \mathcal{F}$, and therefore $\mathcal{X}(x, T) = x\mathcal{X}(1, T)$ for all $x \in \mathbb{R}_+$ and finite stopping times $T$.

1.3. No-short-sale constraints. In real markets, some economic agents, for instance pension funds, face several institution-based constraints when trading. The most important constraint is the admissibility constraint we have introduced: the total wealth of the

\(^1\)Here and in the sequel, any statement involving (in)equalities between processes is understood to hold for all $t \in \mathbb{R}$, $\mathbb{P}$-almost surely; for example, $X \geq 0$ above means $\mathbb{P}[X_t \geq 0, \forall t \in \mathbb{R}_+] = 1$.\]
agent needs to be guaranteed to remain always nonnegative. With Assumption 1.1 in force, and if jumps are potentially present in the asset-price process, in order to ensure nonnegativity of the wealth processes resulting from trading it is both mathematically and economically reasonable to consider the case of no-short-sale constraints in trading.

Fix $T = \{\tau_0 < \tau_1 \ldots < \tau_n\} \in \mathcal{T}$ and consider a strategy described by the predictable process $\theta := \sum^n_{j=1} \vartheta_{\tau_{j-1}} 1_{[\tau_{j-1}, \tau_j]}$, where each $\vartheta_{\tau_{j-1}}$ is $\mathcal{F}_{\tau_{j-1}}$-measurable for $j = 1, \ldots, n$. Define $X^{x, \theta}$ via (1.1) and assume that $X \in \mathcal{X}(x; T)$. In order to ensure that no short sale of the $i$th asset is allowed, we ask that $\theta^i \geq 0$. The amount invested in the baseline asset $S^0$ is $X^{x, \theta} - \sum^d_{i=1} \theta^i S^i$; this has to be nonnegative as well. We therefore define the set $X_\Delta(x; T)$ of all admissible, no-short-sale, buy-and-hold strategies that start from capital $x \in \mathbb{R}_+$ and trade in times included in $T$ to be consisting of those $X^{x, \theta} \in X \in \mathcal{X}(x; T)$ such that $\theta^i \geq 0$ for all $i = 1, \ldots, d$ as well as $\sum^d_{i=1} \theta^i S^i \leq X^{x, \theta}$. This is easily seen to be equivalent to $\vartheta^i_{\tau_{j-1}} \geq 0$ for all $i = 1, \ldots, d$ and $j = 1, \ldots, n$ as well as $\sum^d_{i=1} \vartheta^i_{\tau_{j-1}} S^i \leq X^x_{\tau_{j-1}}$ for all $j = 1, \ldots, n$. The sets $X_\Delta(x, T)$ and $X_\Delta$ are now readily defined. These strategies are those of agents that want to exclude negative total wealth completely.

Remark 1.3. Under reasonable assumptions on jump sizes, restricting attention to no-short-sale strategies is implied by the admissibility requirement. The mathematical details are presented below as full-support condition (FULL-SUPP). The idea is simple: when asset prices can jump upwards in an unbounded manner and downwards arbitrarily close to zero at any time (given that the company is not out of business already), then $X = X_\Delta$.

In other words, admissible strategies necessarily involve no short sales. The full-support condition below roughly states that the log-asset-price returns will be (locally) unbounded both above and below. If one is willing to include jumps in the stochastic modeling of asset-prices, this assumption is perfectly natural: there is no a priori reason why a possible jump in the asset’s log-returns will be bounded above or below. Note that most of the financial-market models including jumps used in practice do satisfy condition (FULL-SUPP).

Let us be a bit more precise now. Remember the definition of the lifetimes $\zeta^i$ from Remark 1.2. For arbitrary stopping times $\tau$ and $\tau'$ with $\mathbb{P}[\tau < \tau'] = 1$, let $\mathbb{R}^I(\tau, \tau')$ denote the (random) subspace of $\mathbb{R}_+$ whose $i$th component is $\mathbb{R}_+$ if $\mathbb{P}[\zeta^i > \tau' \mid \mathcal{F}_\tau] > 0$ and $\{0\}$ if $\mathbb{P}[\zeta^i > \tau' \mid \mathcal{F}_\tau] = 0$. We introduce the following notation: for a random vector $\xi$ and a $\sigma$-field $\mathcal{G} \subseteq \mathcal{F}$, $\overline{\supp} \mathcal{L}(\xi \mid \mathcal{G})$ denotes the closed convex hull of the support of the conditional distribution of $\xi$ given $\mathcal{G}$. We then ask that the following full-support condition is satisfied:

- For arbitrary stopping times $\tau$ and $\tau'$ with $\mathbb{P}[\tau < \tau'] = 1$, we have

\[
\text{(FULL-SUPP)} \quad \overline{\supp} \mathcal{L}(S_{\tau'} \mid \mathcal{F}_\tau) = \mathbb{R}^I(\tau, \tau').
\]

It is then straightforward to deduce that under this condition we have $\mathcal{X}(x; T) = X_\Delta(x; T)$ for all $x \in \mathbb{R}_+$ and planning horizons $T$. Note that if $S$ is one-dimensional ($d = 1$), the full-support condition is equivalent to $S_{\tau'} = 0$ on $\{\mathbb{P}[\zeta^i > \tau' \mid \mathcal{F}_\tau] = 0\}$ and $\mathbb{P}[S_{\tau'} < \epsilon \mid \mathcal{F}_\tau] > 0$ as well as $\mathbb{P}[S_{\tau'} > \epsilon^{-1} \mid \mathcal{F}_\tau] > 0$ on $\{\mathbb{P}[\zeta^i > \tau' \mid \mathcal{F}_\tau] > 0\}$ for all $\epsilon > 0$. 

2. Unbounded Profits with Bounded Risks, Supermartingale Deflators, and the Semimartingale Property of Asset-Price Processes

2.1. Unbounded Profit with Bounded Risk. We define here a rather weak “no-free-lunch” concept that will be of major importance in our discussion. It is a weakened version of the “No Free Lunch with Vanishing Risk” (NFLVR) condition introduced in [DS94]. More precisely, the following definition represents one of the two parts that comprise the NFLVR condition; the other part is the classical “No Arbitrage” condition.

Definition 2.1. A market where only no-short-sale, buy-and-hold trading is allowed satisfies the no unbounded profit with bounded risk (NUPBR) condition if for all \( x \in \mathbb{R}^+ \) and \( T \in \mathbb{R}^+ \), the collection \( X_{\Delta}(x, T) \) is bounded in probability, i.e.,

\[
\lim_{a \to \infty} \sup_{X \in X_{\Delta}(x, T)} \mathbb{P}[X_T > a] = 0.
\]

Since \( X_{\Delta}(x, T) = xX_{\Delta}(1, T) \), we only have to check the above condition for \( x = 1 \). Note also that if (2.1) is valid for \( T \in \mathbb{R}^+ \), it also holds for all finite stopping times \( T \).

If condition NUPBR\( \Delta \) fails, one can find some financial planning horizon \( T \), a sequence \( (X^k)_{k \in \mathbb{N}} \) of elements in \( X_{\Delta}(1, T) \) and a \( p > 0 \) such that \( \mathbb{P}[X^k_T > k] > p \) for all \( k \in \mathbb{N} \). This sequence \( (X^k)_{k \in \mathbb{N}} \) has bounded risk, that is, no more than unit losses, while with at least some fixed positive probability \( p > 0 \) can make unbounded profit, which explains the appellation of the condition in Definition 2.1.

One can also naturally define the NUPBR condition (there is no use of the subscript “\( \Delta \)” now) when using any buy-and-hold admissible processes, replacing the sets “\( X_{\Delta}(x, T) \)” in Definition 2.1 with “\( X(x, T) \)”.

2.2. Supermartingale deflators. We now introduce a concept that is closely related to that of equivalent (super)martingale probability measures, but weaker. It appears as the natural dual domain in the solution of the utility maximization problem from terminal wealth in [KS99], as well as in [KK07] in a context close to what will be discussed in this section, but in a general semimartingale setting and using continuous trading.

Definition 2.2. The class of supermartingale deflators for no-short-sale, buy-and-hold trading is defined as

\[
\mathcal{Y}_\Delta := \{ Y > 0 \mid Y_0 = 1, \text{ and } YX \text{ is a supermartingale for all } X \in \mathcal{X}_\Delta \}.
\]

If a supermartingale deflator \( Y \) exists, condition NUPBR\( \Delta \) holds. Indeed, for all finite stopping times \( T \), we have \( \sup_{X \in X_{\Delta}(x, T)} \mathbb{E}[Y_TX_T] \leq x \); this implies in particular that \( (Y_TX_T)_{X \in X_{\Delta}(x, T)} \) is bounded in probability, and since \( \mathbb{P}[Y_T > 0] = 1 \) we have that \( X_T \) is bounded in probability as well.

Under Assumption 1.1 and \( \mathcal{Y}_\Delta \neq \emptyset \) we now show that \( S \) is a semimartingale. If \( Y \in \mathcal{Y}_\Delta \), then \( Y \) is a strictly positive supermartingale meaning that \( \mathbb{P}[\inf_{t \in [0, T]} Y_t > 0] = 1 \) for all \( T > 0 \); therefore \( 1/Y \) is a semimartingale. Since for all \( i = 1, \ldots, d \), \( S^i \in \mathcal{X}_\Delta \), it follows
that $YS^i$ is a supermartingale, thus a semimartingale. Finally, $S^i = (1/Y)YS^i$ is a semimartingale for all $i = 1, \ldots, d$, which completes the argument.

The set $\mathcal{Y}$ (involving no subscript “$\Delta$” now) is defined similarly to $\mathcal{Y}_\Delta$, replacing “$\mathcal{X}_\Delta$” in Definition 2.2 with “$\mathcal{X}$”. Obviously, $\mathcal{Y} \subseteq \mathcal{Y}_\Delta$. If $\mathcal{Y} \neq \emptyset$ then NUPBR holds and $S$ is a semimartingale — the proof of these claims is identical as for the no-short-sale case.

2.3. A weak version of the Fundamental Theorem of Asset Pricing. In the previous Subsection 2.2 we have seen some connection between the concepts of NUPBR$\Delta$, supermartingale deflators and the semimartingale property of $S$. These are immensely tied to each other, as is now revealed.

**Theorem 2.3.** Given Assumption 1.1 on $S$, the following are equivalent:

1. The NUPBR$\Delta$ condition holds.
2. $\mathcal{Y}_\Delta \neq \emptyset$.
3. $S$ is a semimartingale.

Proofs of (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) in Theorem 2.3 above have been discussed in Subsection 2.2. The implications (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2), though not obvious, follow from more general considerations contained in [KK07], see Subsection 6.1 for slightly more details. The proof of the most interesting implication (1) $\Rightarrow$ (3) is contained in §6.1.1.

2.4. Remarks on Theorem 2.3. Before moving on, we comment on some aspects regarding the implications of Theorem 2.3 above.

2.4.1. **Modeling without semimartingales.** Implication (1) $\Rightarrow$ (3) in Theorem 2.3 excludes in our natural setting asset-price processes that are not semimartingales. This means, it does not permit models involving fractional Brownian motion with statistical dependence of increments. One needs, for instance, the introduction of frictions in a fractional Brownian motion setting to establish a reasonable financial market model — see for example [Gua06] for a situation where viability of market models that include fractional Brownian motion is achieved in the presence of proportional transaction costs.

2.4.2. **Possibility of unbounded profits if more freedom is allowed.** The NUPBR$\Delta$ condition does not imply the more restrictive non-constrained NUPBR version. This can be easily seen by taking $S$ to be any deterministic continuous increasing function with $S_T > S_0$ for some $T > 0$. Then, $n(S - S_0) \in \mathcal{X}(0, T)$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} n(S_T - S_0) = \infty$, contradicting the NUPBR condition.

One should note that all conditions in Theorem 2.3 are extremely weak. Further reasonable economic considerations should lead to appropriately tailored and more detailed models than, for example, the one described in the previous paragraph.
2.4.3. **Comparison with the work of Delbaen and Schachermayer.** Theorem 7.2 of the seminal paper [DS94] establishes the semimartingale property of $S$ under the NFLVR condition for buy-and-hold strategies, coupled with a local boundedness assumption (together with the càdlàg property and $\mathbf{F}$-adaptedness, of course) on $S$.

The assumptions in implication (1) $\Rightarrow$ (3) of Theorem 2.3 are weaker than the ones in [DS94]. Condition NUPBR$_\Delta$ is weaker than NUPBR, which in turn is even weaker than NFLVR for buy-and-hold strategies. Furthermore, local boundedness from above is not required in our context and positivity of the asset-price processes is not essential. All that is required is local boundedness from below, as is explained in §2.4.4. Therefore, the statement of Theorem 2.3 is more general than Theorem 7.2 in [DS94]. We note that if $S$ is unbounded both from above and below, the implication (1) $\Rightarrow$ (3) of Theorem 2.3 is no longer necessarily true; see Example 7.5 in [DS94].

The alternative proof of implication (1) $\Rightarrow$ (3) of Theorem 2.3 that is provided in §6.1.1, most importantly, does not use the deep Bichteler-Delacherie theorem on the characterization of semimartingales as “good integrators”, see for example [Bic02, Pro05], where one starts by defining semimartingales as good integrators and gets the classical definition as a byproduct. Our result can be seen as a “multiplicative” counterpart of the Bichteler-Delacherie theorem, and its proof exploits two simple facts: (a) positive supermartingales are semimartingales, a statement that follows directly from the Doob-Meyer decomposition theorem; and (b) reciprocals of strictly positive supermartingales are semimartingales, which is a consequence of Itô’s formula. Crucial in the proof are also the concepts of supermartingale deflators and the numéraire portfolio. The numéraire portfolio is in some sense the “best” performing admissible wealth process which makes all other admissible wealth processes behave as supermartingales, when discounted by it.

2.4.4. **A generalization.** Though most interesting from a mathematical rather than economical viewpoint, Theorem 2.3 is valid even without the nonnegativity of the asset-price processes stated in Assumption 1.1 — all that is required is local boundedness from below. Specifically, implication (1) $\Rightarrow$ (3) of Theorem 2.3 will be proved in §6.1.1 under the assumption that the process $S$ is càdlàg, $\mathbf{F}$-adapted, and that there exists an increasing sequence $(t_m)_{m \in \mathbb{N}}$ of stopping times with $\lim_{m \to \infty} t_m = +\infty$ such that $\inf_{t \in [0,t_m]} S_t^i > -m$ for all $i = 1, \ldots, d$.

Furthermore, with an appropriate relaxation of the assumption on the constraints, we can get a stronger NUPBR condition that will be equivalent to the conditions of Theorem 2.3, as we now describe. Let $\mathcal{C}$ be a compact $\mathbb{R}^d$-set valued process; this means that $\mathcal{C}(\omega, t)$ is a compact subset of $\mathbb{R}^d$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. For some $\mathcal{T} \subseteq \mathcal{F}$ and $x \in \mathbb{R}_+$, call $\mathcal{X}_\mathcal{C}(x, \mathcal{T})$ the set of all elements $X^{x, \theta} \in \mathcal{X}(x; \mathcal{T})$ that satisfy $\theta(\omega, t) \in X^{x, \theta}(\omega, t)\mathcal{C}(\omega, t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. The case of no-short-sale constraints under Assumption 1.1 corresponds to $\mathcal{C} \equiv \{ x \in \mathbb{R}^d \mid x^i \geq 0, \text{ for all } i = 1, \ldots, d, \text{ and } \sum_{i=1}^d x^i \leq 1 \}$. The NUPBR$_\mathcal{C}$ condition is now obviously defined, as is the set of supermartingale deflators $\mathcal{Y}_\mathcal{C}$.
We now give a bit more structure to $\mathcal{C}$, assuming that it is an $\mathbf{F}$-adapted, càdlàg process such that $0 \in \mathcal{C}(\omega, t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. Adaptedness means that for all compact sets $F \subseteq \mathbb{R}^d$ we have $\{\mathcal{C}_t \cap F \neq \emptyset\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}_+$. The càdlàg property is defined as usual, using the natural Hausdorff metric topology on the class of compact subsets of $\mathbb{R}^d$. Under these conditions, and under the assumption that $S$ is any semimartingale, we have $Y \neq \emptyset$ and therefore condition NUPBR as well. Proof of this is the content of §6.1.1.

3. Continuous-Path Asset-Price Processes and Non-Constrained Trading

3.1. Set-up and some notation. For this section we shall be assuming that $S$ is a continuous-path, $\mathbf{F}$-adapted process. The nonnegativity Assumption 1.1 will not be in force. Again, it is not assumed that $S$ is a semimartingale, but this will be a consequence of Theorem 3.1 below. Note that if $S$ is a semimartingale, then one has the decomposition $S = A + M$, where $A = (A^1, \ldots, A^d)$ has continuous paths and is of finite variation, and $M = (M^1, \ldots, M^d)$ is a continuous local martingale. Denote by $[M^i, M^k]$ the quadratic (co)variation of $M^i$ and $M^k$. Also, let $[M, M]$ be the $d \times d$ nonnegative-definite symmetric matrix-valued process whose $(i, k)$-component is $[M^i, M^k]$. Call now $G := \text{trace}[M, M]$, where trace is the operator returning the trace of a matrix. Observe that $G$ is an increasing, adapted, continuous process and that there exists a $d \times d$ nonnegative-definite symmetric matrix-valued process $c$ such that $[M^i, M^k] = \int_0^\cdot c^{i,k}_t \mathrm{d}G_t$; $[M, M] = \int_0^\cdot c_t \mathrm{d}G_t$ in short.

3.2. Another version of the Fundamental Theorem of Asset Pricing. In accordance to Theorem 2.3, we now have the following result.

**Theorem 3.1.** In a market with continuous asset-price processes $S$ where only buy-and-hold trading is allowed, the following are equivalent:

1. The NUPBR condition holds.
2. A supermartingale deflator exists: $Y \neq \emptyset$.
3. The price-process $S$ is a semimartingale. Writing its Doob-Meyer decomposition $S = A + M$ with $[M, M] = \int_0^\cdot c_t \mathrm{d}G_t$ as above, there exists a $d$-dimensional, predictable process $\rho$ such that $A = \int_0^T (c_t \rho_t) \mathrm{d}G_t$ and $\int_0^T \langle \rho_t, c_t \rho_t \rangle \mathrm{d}G_t < \infty$ for all $T \in \mathbb{R}_+$.

The proof of (2) $\Rightarrow$ (1) was already discussed in Subsection 2.2. Proving (3) $\Rightarrow$ (2) is straightforward, see §3.3.2 below. Again, it is the implication (1) $\Rightarrow$ (3) that is the most intricate; we provide a full proof of this in Subsection 6.2 that only uses the fact that continuous local martingales are time-changed Brownian motions.

The statement of Theorem 3.1 resembles very closely its counterpart statement of Theorem 2.3. The difference lies in condition (3), which here appears to be more restrictive. The fact that the equivalent condition in Theorem 2.3 is simpler is a consequence of the constraints that were enforced on strategies. Existence of a predictable $d$-dimensional process $\rho$ such that $A = \int_0^\cdot (c_t \rho_t) \mathrm{d}G_t$ in condition (3) of Theorem 3.1 turns out to be equivalent to nonexistence of wealth strategies starting from zero initial capital, staying
nonnegative at all times, and managing to escape zero with positive probability. Existence of such strategies is certainly possible to assume, but may require continuous trading to be utilized. Given the existence of such a predictable process \( \rho \), \( \int_0^T \langle \rho_t, c_t \rho_t \rangle \, dG_t < \infty \) for all \( T \in \mathbb{R}_+ \) will always hold if \( \rho \) is (locally) bounded. The only way for this to fail is if \( \rho \) can become possibly unbounded; then, one could construct an increasing profit essentially following the vector process \( \rho \) as being the percentage of wealth invested in each asset at every time. Since \( \rho \) is unbounded, this will require eventually immense short-sales to be implemented, which is questionable from a practical point of view. In this sense, the statement of Theorem 3.1, although clear and elegant from a mathematical point of view, has practical limitations if used as a guideline for modeling financial markets.

3.3. Remarks on Theorem 3.1. The remarks below made on Theorem 3.1 pertain only to the case of continuous-path asset-price processes under no trading constraints.

3.3.1. Market price of risk and the numéraire portfolio. Condition (3) of Theorem 3.1 has some economic consequences. Assume for simplicity that \( G \) is absolutely continuous with respect to Lebesgue measure, i.e., that \( G := \int_0^t g_s \, ds \) for some predictable process \( g \). In this case, take \( c^{1/2} \) to be any root of the nonnegative-definite matrix \( c \) (that can be chosen in a predictable way) and define \( \sigma := c^{1/2} \sqrt{g} \). Then, we can write \( dS_t = \sigma_t (\lambda_t \, dt + dW_t) \), where \( W \) is a standard \( d \)-dimensional Brownian motion\(^2\) and \( \lambda \) is the canonical market price of risk process (in the one-dimensional case also commonly known as the Sharpe ratio), that has to satisfy \( \int_0^T |\lambda_t|^2 \, dt < \infty \) for all \( T \in \mathbb{R}_+ \). We conclude that the NUPBR condition holds if and only if a market-price-of-risk process exists and is locally square-integrable in a pathwise sense. Note that the market-price-of-risk process \( \lambda \) is exactly the volatility of the wealth process generated by the numéraire portfolio.

3.3.2. Local martingale deflators. A quick proof of the implication (3) \( \Rightarrow \) (2) of Theorem 3.1 will be now provided. With the data of condition (3) there, define the process
\[
\tilde{Y} := \exp \left( - \int_0^t \langle \rho_s, dS_t \rangle + \frac{1}{2} \int_0^t \langle \rho_s, c_s \rho_t \rangle \, dG_t \right).
\]
Condition (3) ensures that \( \tilde{Y} \) is well-defined (meaning that the two integrals above make sense); a simple use of integration-by-parts gives that \( \tilde{Y} S^i \) is a local martingale for all \( i = 0, \ldots, d \). This in turn, using integration-by-parts again, implies that \( \tilde{Y} X \) is a local martingale for all \( X \in \mathcal{X} \), which is a stronger statement than \( \mathcal{Y} \neq \emptyset \). It follows that the condition \( \mathcal{Y} \neq \emptyset \) implies the existence of a local martingale deflator, a concept that in [SY98] is coined strict martingale density. In fact, the special structure of continuous semimartingales will imply that any element \( Y \in \mathcal{Y} \) can be uniquely decomposed as \( Y = \tilde{Y} N B \), where \( N \) is a strictly positive local martingale with \( N_0 = 1 \) that is strongly

\(^2\)In the case where \( c \) is nonsingular for Lebesgue-almost every \( t \in \mathbb{R} \), \( \mathbb{P} \)-almost surely, we have \( W := \int_0^t c_t^{-1/2} \, dB_t \). If \( c \) fails to be nonsingular for Lebesgue-almost every \( t \in \mathbb{R} \), \( \mathbb{P} \)-almost surely, one can still construct a Brownian motion \( W \) in order to have \( M = \int_0^t c_t^{1/2} \, dW_t \), holding by enlarging the probability space — check for example [KS91], Theorem 4.2 of Section 3.4.
orthogonal to $S$ in the sense that $[N,S] = 0$ and $B$ is a strictly positive decreasing process with $B_0 = 1$. The maximal elements of $\mathcal{Y}$ are of course those that satisfy $B \equiv 1$ and are all local martingale deflators. As a final remark, note that $1/\tilde{Y}$ is the numéraire portfolio when continuous, non-constrained trading is allowed.

4. Utility Maximization

The purpose of this section is to show that for utility-maximizing economic agents, allowing only buy-and-hold trading does result (given appropriately high trading frequency) in optimal utilities and wealth processes as close as desired to their theoretical continuous-trading optimal counterparts.

4.1. Trading in continuous time. Theorems 2.3 and 3.1 bring forth semimartingales in financial modeling, and also the use of stochastic integration with respect to predictable processes, not necessarily of the simple buy-and-hold structure we have been discussing up to now.

Let us introduce some notation to be used below. If $S$ is a semimartingale, $X(x,T)$ will denote the class of all admissible (meaning, nonnegative) processes that can be achieved starting from $x$, having financial planning horizon equal to $T$ and trading using any predictable process that vanishes outside $[0,T]$; obviously $X(x,T) \subseteq X(x,T)$. We define also the corresponding class $\bar{X}$ of all possible admissible wealth processes. Furthermore, $\bar{X}_\Delta(x,T)$ will be the subset of $\bar{X}(x,T)$ consisting of no-short-sale continuous-trading strategies; $\bar{X}_\Delta$ is then defined in the obvious way.

4.2. The utility maximization problem. A utility function is an increasing and concave function $U : (0,\infty) \mapsto \mathbb{R}$. We extend the definition to cover zero wealth via $U(0) := \lim_{x \downarrow 0} U(x)$. Note that no regularity conditions are imposed on $U$.

For $T \in \mathcal{T}$ and $x \in \mathbb{R}_+$, call $T := \sup \mathcal{T}$ and define the agent-specific indirect utility

$$u_\Delta(x;T) := \sup_{X \in \mathcal{X}_\Delta(x;\mathcal{T})} \mathbb{E}[U(X_T)].$$

It is obvious that for all $T \in \mathcal{T}$, $u_\Delta(\cdot;T) \leq \infty$ for some $x > 0$ if and only $u_\Delta(x;T) < \infty$ for all $x \in \mathbb{R}$. In particular, if $u_\Delta(x;\mathcal{T}) < \infty$ for some $x > 0$, $u_\Delta(\cdot;\mathcal{T})$ is a proper continuous concave function. If $U$ is strictly concave (in which case it is a fortiori strictly increasing as well) and a solution to the utility maximization problem defined above exists, it is necessarily unique.

Define also the maximal indirect utility that can be achieved via no-short-sale buy-and-hold strategies for all $x \in \mathbb{R}$ and financial planning horizons $T$ via

$$u_\Delta(x,T) := \sup_{T \in \mathcal{T}} u_\Delta(x;T).$$

It is easy to see that $u_\Delta(\cdot, T)$ is a concave function for all finite stopping times $T$, but we shall have a lot more to say in Theorem 4.1 below.
Finally, define the indirect utility when continuous trading is allowed via

\[ u_\Delta(x, T) := \sup_{X \in \mathcal{X}(x, T)} \mathbb{E}[U(X_T)]. \]

It is obvious that \( u_\Delta \leq \overline{u}_\Delta \).

4.3. Near-optimality using buy-and-hold strategies. The aim of the next result is to show that the value functions \( u \) and \( \overline{u} \) are actually equal and that “near optimal” wealth processes for the buy-and-hold case approximate arbitrarily close the solution of the continuous trading case, if the latter exists.

**Theorem 4.1.** In what follows, except the last statement (4), the asset-price process \( S \) is assumed to satisfy Assumption 1.1. Using all notation introduced above, we have:

1. \( u_\Delta(x, T) = \overline{u}_\Delta(x, T) \) for all \( x \in \mathbb{R}_+ \) and financial planning horizons \( T \).
2. Suppose that NUPBR\(_\Delta\) holds, \( U \) is strictly concave, and \( u_\Delta(\cdot, T) < \infty \) for some financial planning horizon \( T \). Then, for any \( x \in \mathbb{R}_+ \), any \( \mathcal{X}(x, T) \)-valued sequence \((X^n)_{n \in \mathbb{N}}\) and any \( \mathcal{X}(x, T) \)-valued sequence \((\overline{X}^n)_{n \in \mathbb{N}}\) with \( \lim_{n \to \infty} \mathbb{E}[U(X^n_T)] = u_\Delta(x, T) = \overline{u}_\Delta(x, T) = \lim_{n \to \infty} \mathbb{E}[U(\overline{X}^n_T)] \), we have \( \mathbb{P}\lim_{n \to \infty} |X^n_T - \overline{X}^n_T| = 0 \).
3. Suppose that \( U \) is strictly concave and that for some \( x \in \mathbb{R}_+ \) and financial planning horizon \( T \) there exists \( \overline{X} \in \mathcal{X}(x, T) \) with \( \overline{X} > 0 \) and \( \mathbb{E}[U(\overline{X}_T)] = \overline{u}_\Delta(x, T) < \infty \). Then, for any \( \mathcal{X}(x, T) \)-valued sequence \((X^n)_{n \in \mathbb{N}}\) such that \( \lim_{n \to \infty} \mathbb{E}[U(X^n_T)] = u_\Delta(x, T) \) we have \( \mathbb{P}\lim_{n \to \infty} \sup_{t \in [0, T]} |X^n_t - \overline{X}_t| = 0 \).
4. If \( S \) is continuous, all of the above statements (1), (2) and (3) above also hold when we consider non-constrained admissible wealth processes, simply removing all subscripts “\( \Delta \)” from the wealth process sets and the indirect utility functions.

4.4. Remarks on Theorem 4.1.

4.4.1. The utility maximization problem for continuous trading has attracted a lot of attention and has been successfully solved using convex duality methods. In particular, in [KS03] the authors show that an optimal solution (wealth process) to problem (4.1) exists for all \( x \in \mathbb{R}_+ \) and fixed financial planning horizon \( T \) under the following conditions: \( U \) is strictly concave and continuously differentiable in \((0, \infty)\), satisfies the Inada conditions \( \lim_{x \downarrow 0} U'(x) = +\infty \), \( \lim_{x \uparrow +\infty} U'(x) = 0 \), as well as the finite dual value function condition\(^3\) \( \inf_{y \in \mathcal{Y}} \mathbb{E}[V(yY_T)] < +\infty \) holds for all \( y \in (0, \infty) \), where \( V \) is defined to be the Legendre-Fenchel transform of \( U \), i.e., \( V(y) := \sup_{x > 0} \{U(x) - xy\} \). The above conditions can be used to ensure existence of the optimal wealth process in statement (3) of Theorem 4.1.

\(^3\)It is tacitly assumed here that \( \mathcal{Y} \neq \emptyset \), i.e., that NUPBR holds. The authors of [KS03] prove existence of optimal solutions to the utility maximization problem under the stronger NFLVR condition (which is equivalent to the existence of some \( Y \in \mathcal{Y} \) with \( \mathbb{E}[Y_T] = 1 \) — existence of an equivalent local martingale measure), but all that is really needed is \( \mathcal{Y} \neq \emptyset \).
4.4.2. Observe that in statement (1), condition NUPBR\(\triangle\) is not needed.

In statement (2), neither strict concavity nor the condition NUPBR\(\triangle\) can be dispensed in order to get the result, as we briefly discuss now. In cases where the supremum in \(u\(\triangle\)(x, T)\) is attained, in absence of strict concavity the optimum is not necessarily unique. Further, if NUPBR\(\triangle\) fails one can find wealth processes \(X^n \in \mathcal{A}(1, T)\), for some \(T \in \mathbb{R}_+\), such that \(\mathbb{P}\)-\(\lim_{n \to \infty} X^n_T = \infty\) on some event \(A\) with \(\mathbb{P}[A] > 0\).

Finally, even though statement (3) does not directly assume the condition NUPBR\(\triangle\), it is indirectly in force because of the existence of \(X \in \mathcal{A}(x, T)\) with \(X > 0\) and \(\mathbb{E}[U(X_T)] = u\(\triangle\)(x, T) < \infty\). For more information, see Proposition 4.19 in [KK07].

4.4.3. The difference between statements (2) and (3) in Theorem 4.1 is that in the latter case we can infer uniform convergence of the wealth processes to the limiting one, while in the former we only have convergence of the terminal wealths. It is an open question whether the uniform convergence of the wealth processes can be established without assuming that the utility maximization problem involving continuous trading has a solution.

4.4.4. The assumption that \(U\) is increasing can be dropped from statements (1) and (2) of Theorem 4.1, if one makes instead the mild assumption that \(S\) is locally bounded. We do not go into details, since it is more of a mathematical, and less of an economical, value.

5. Approximating Positive Stochastic Integrals via Simple Integration

This whole section is devoted to proving Theorem 5.1 below, which in effect is an approximation result of wealth processes obtained from continuous trading via buy-and-hold strategies. Theorem 5.1, interesting in its own right, will prove essential in proving Theorem 4.1.

All notation from the main text is kept. We define \(\mathcal{A}(x)\) to be the union of \(\mathcal{A}(x; T)\) for all finite stopping times \(T\); \(\overline{\mathcal{A}}(x)\) is defined similarly. In this and the next section convergence of processes in probability uniformly on compact sets of the real line will be considered; \(\text{ucP}-\lim_{n \to \infty} \xi^n = \xi\) means that, for all \(T \in \mathbb{R}\), \(\mathbb{P}-\lim_{n \to \infty} \sup_{t \in [0, T]} |\xi^n_t - \xi_t| = 0\). Note that \text{ucP}-convergence comes from a metric topology.

A version of Theorem 5.1 below can also be found in [Str03], where the author uses it to approximate the optimal wealth process for the exponential utility maximization problem via buy-and-hold strategies.

**Theorem 5.1.** Assume that \(S\) is a \(d\)-dimensional semimartingale.

1. If \(S\) is continuous then for all \(X \in \overline{\mathcal{A}}(x)\) there exists a \(\mathcal{A}(x)\)-valued sequence \((X^k)_{k \in \mathbb{N}}\) such that \(\text{ucP}-\lim_{k \to \infty} X^k = X\).

2. If \(S\) satisfies Assumption 1.1, then for all \(X \in \overline{\mathcal{A}}(x)\) there exists a \(\mathcal{A}(x)\)-valued sequence \((X^k)_{k \in \mathbb{N}}\) such that \(\text{ucP}-\lim_{k \to \infty} X^k = X\).

If there further exists some \(\epsilon > 0\) such that \(X \geq \epsilon\), the aforementioned approximating sequences can be chosen in a way that \(X^k \geq \epsilon\) for all \(k \in \mathbb{N}\).
The proof of Theorem 5.1 will be given treating the continuous and discontinuous cases separately. The special structure of continuous-path processes give way to an “additive” approximation of the stochastic integrals; in the presence of jumps, this will not work any more and one has to work harder and obtain some “multiplicative” approximation, which also makes more sense from a trading viewpoint.

Before we delve into the proofs, let us quickly comment that the last statement of Theorem 5.1 is almost trivial once the claims preceding it have been proved. We only discuss the case described in statement (1) — the case of statement (2) is treated mutatis mutandis. If \( \mathcal{X}(x) \ni X \geq \epsilon \) then \( (X - \epsilon) \in \mathcal{X}(x - \epsilon) \); this means that we can find some \( \mathcal{X}(x - \epsilon) \)-valued sequence \( (\xi^k)_{k \in \mathbb{N}} \) such that \( \text{ucP-lim}_{k \to \infty} \xi^k = X - \epsilon \). Then, \( X^k := \epsilon + \xi^k \) satisfies \( X^k \in \mathcal{X}(x) \) and \( X^k \geq \epsilon \) for all \( k \in \mathbb{N} \), as well as \( \text{ucP-lim}_{k \to \infty} X^k = X \).

In order to avoid cumbersome notation, from here onwards the dot “.” between two processes will denote stochastic integration.

### 5.1. Proof of statement (1) of Theorem 5.1.

Write the Doob-Meyer decomposition \( S = A + M \), where \( A \) is a process of finite variation and \( M \) a continuous local martingale.

Consider \( X^{x,\theta} \in \mathcal{X}(x) \) for some predictable \( d \)-dimensional process \( \theta \). For \( a \in \mathbb{R}_+ \), let \( \tau_a := \inf\{ t \in \mathbb{R}_+ \mid x + (\theta^a, \theta^a) \cdot S_t = 0 \} \) and define \( \theta^a := \theta^a t \). We have \( X^{x,\theta^a} \in \mathcal{X}(x) \) for all \( a > 0 \) and it is straightforward to check that \( \text{ucP-lim}_{a \to 0} X^{x,\theta^a} = X^{x,\theta} \). In other words, we can assume without loss of generality that \( X^{x,\theta} \in \mathcal{X}(x) \) is such that \( |\theta| \leq a \) for some \( a > 0 \). Further, via a localization argument we may suppose that \( \int_0^T |dA_t| \) and \( [M, M]_T \) are bounded. By an easy density argument then we get the existence of simple integrands \( (\eta^k)_{k \in \mathbb{N}} \) such that \( \int_0^T |\eta^k - \theta| |dA_t| \) and \( [(\eta^k - \theta) \cdot M, (\eta^k - \theta) \cdot M]_T \) \( \mathbb{P} \)-converge to zero. It now easily follows that \( \mathbb{P} \)-\( \text{lim}_{k \to \infty} \sup_{t \in [0, T]} |X^{x,\eta^k}_t - X^{x,\theta}_t| = 0 \).

### 5.2. Proportional trading.

Sometimes it pays off more to regard investment in relative, rather than absolute terms. This means looking at the percentage of current wealth invested in some asset rather than units of the asset held in the portfolio.

If \( S \) is a semimartingale satisfying Assumption 1.1, we consider the total return process \( R = (R^1, \ldots, R^d) \), where \( R \) satisfies \( R_0 = 0 \) and the system of stochastic differential equation \( dS^i_t = S^i_t dR^i_t \) for \( i = 1, \ldots, d \) and \( t \in \mathbb{R} \). In other words, \( S^i = S^i_0 \mathcal{E}(R^i) \), where \( \mathcal{E} \) is the stochastic exponential operator. It should be noted that, for \( i = 1, \ldots, d \), the process \( R^i \) only lives in the stochastic interval \( [0, \zeta^i] \) for the lifetimes \( \zeta^i \) defined in Remark 1.2, and that it might explode at time \( \zeta^i \). However, it is easy to see that this does not affect the validity of the conclusions below.

Define now the closed \( d \)-dimensional simplex

\[
\Delta^d := \left\{ x \in \mathbb{R}^d \mid x^i \geq 0 \text{ for all } i = 1, \ldots, d, \text{ and } \sum_{i=1}^d x^i \leq 1 \right\}.
\]

For any predictable \( \Delta^d \)-valued process \( \pi \), consider the wealth process \( X^{(x, \pi)} \) defined via

\[
X^{(x, \pi)} := x \mathcal{E} \left( \int_0^\cdot \langle \pi_t, dR_t \rangle \right).
\]
Observe that we are using parentheses in the \((x, \pi)\) superscript of \(X\) in (5.2) to distinguish from a wealth process of the form \(X^{x, \theta} = x + \theta \cdot X\), generated by \(\theta\) in the additive way.

Considering \(X^{(x, \pi)}\) when ranging \(\pi\) over all the predictable \(\overline{\Delta}^d\)-valued processes that vanish outside of \([0, T]\) gives us the whole class \(\mathcal{X}(x, T)\).

5.3. Integral approximation in a multiplicative way. Start with some predictable càglàd (left continuous with right limits) and adapted, thus predictable, \(\overline{\Delta}^d\)-valued process \(\pi\). The wealth process generated by \(\pi\) in a multiplicative way starting from \(x \in \mathbb{R}_+\) is \(X^{(x, \pi)}\), as defined in (5.2).

Consider now some economic agent who may only invest in times included in \(\mathbb{T} = \{\tau_0 < \tau_1 < \ldots < \tau_n\}\). Wanting to follow \(X^{(x, \pi)}\) closely, the agent will decide at each possible trading instant to rearrange the portfolio wealth in such a way as to follow proportional investment. More precisely, the agent will rearrange wealth at time \(\tau_{j-1}, j = 1, \ldots, n\), in a way such that a proportion \(\pi^i_{\tau_{j-1}+} := \lim_{\tau \downarrow \tau_{j-1}} \pi^i_{\tau}\) is held in the \(i\)th asset, \(i = 1, \ldots, d\). Starting from initial capital \(x \in \mathbb{R}_+\) and following the above-described strategy, the agent’s wealth always remains nonnegative and, at time \(t \in \mathbb{R}_+\), is given by

\[
X^{(x, \pi; \mathbb{T})}_t := x \prod_{j=1}^n \left\{ 1 + \sum_{i=1}^d \pi^i_{\tau_{j-1}+} \left( \frac{S^{i}_{\tau_{j-1} \wedge t} - S^{i}_{\tau_{j-1} - 1 \wedge t}}{S^{i}_{\tau_{j-1} - 1 \wedge t}} \right) \right\}.
\]

We note that, for all \(i = 1, \ldots, d, j = 1, \ldots, n\) and \(t \in \mathbb{R}_+\), the ratio \((S^{i}_{\tau_{j-1} \wedge t} - S^{i}_{\tau_{j-1} - 1 \wedge t})/S^{i}_{\tau_{j-1} - 1 \wedge t}\) is assumed to be zero on the event \(\{S^{i}_{\tau_{j-1} \wedge t} = 0\}\). It is obvious that \(X^{(x, \pi; \mathbb{T})} \in \mathcal{X}(x, \mathbb{T})\).

Take a sequence \((\mathbb{T}^k)_{k \in \mathbb{N}}\) in \(\mathfrak{T}\) and write \(\mathbb{T}^k \equiv \{\tau^k_0 < \ldots < \tau^k_n\}\). We say that \((\mathbb{T}^k)_{k \in \mathbb{N}}\) converges to the identity if \(\lim_{k \to \infty} \tau^k_n = \infty\), as well as \(\sup_{j=1, \ldots, n} |\tau^k_j - \tau^k_{j-1}| = 0\), with convergence happening \(\mathbb{P}\)-a.s. in both cases.

**Theorem 5.2.** Assume that \(S\) is a \(d\)-dimensional semimartingale satisfying Assumption 1.1. Consider any predictable càglàd and adapted \(\overline{\Delta}^d\)-valued process \(\pi\). If the \(\mathfrak{T}\)-valued sequence \((\mathbb{T}^k)_{k \in \mathbb{N}}\) converges to the identity, we have \(\text{ucP-lim}_{k \to \infty} X^{(x, \pi; \mathbb{T}^k)} = X^{(x, \pi)}\).

**Proof.** It is easy to see that \(\text{ucP-lim}_{k \to 0} X^{(x, (1-\epsilon)\pi)} = X^{(x, \pi)}\), as well as that, for all \(T \in \mathfrak{T}\), \(\text{ucP-lim}_{k \to 0} X^{(x, (1-\epsilon)\pi; \mathbb{T})} = X^{(x, \pi; \mathbb{T})}\). It then follows that we might assume that \(\pi\) is actually \((1 - \epsilon)\overline{\Delta}^d\)-valued, which means that \(X^{(x, \pi)}\), as well as \(X^{(x, \pi, \mathbb{T}^k)}\) for all \(k \in \mathbb{N}\), remain strictly positive. Actually, since the jumps in the returns of the wealth processes involved are bounded below by \(1 - \epsilon\), the wealth processes themselves are bounded away from zero in compact time-intervals, with the strictly positive bound possibly depending on the path. It then follows that \(\text{ucP-lim}_{k \to \infty} X^{(x, \pi; \mathbb{T}^k)} = X^{(x, \pi)}\) is equivalent to \(\text{ucP-lim}_{k \to \infty} \log X^{(x, \pi; \mathbb{T}^k)} = \log X^{(x, \pi)}\), which is what shall be proved. To ease notation in the course of the proof we shall assume that \(d = 1\). This is done only for typographical convenience; one can read the whole proof for the case of \(d\) assets, if multiplication and division of \(d\)-dimensional vectors are understood in a coordinate-wise sense.
To proceed with the proof, write
\begin{equation}
(5.4) \log \frac{X^{(x,\pi, \mathbb{T}^k)}}{X^{(x,\pi)}} = \sum_{j=1}^{k} \log \left( 1 + \pi_{j+1} \frac{S_{\tau_{j+1}^k} - S_{\tau_{j}^k}}{\Delta \tau_{j}^k} \right) - \left( \pi \cdot R - \frac{1}{2}[\pi \cdot R^c, \pi \cdot R^c] - \sum_{t \leq \tau_k} (\pi_t \Delta R_t - \log (1 + \pi_t \Delta R_t)) \right),
\end{equation}
where $R^c$ is the uniquely-defined continuous local martingale part of the semimartingale $R$. Define the càglàd, predictable process $\eta := (\pi/S) \mathbb{1}_{\{S > 0\}}$. For $k \in \mathbb{N}$ and $j = 1, \ldots, k$, define $\Delta_j^k := S_{\tau_j^k} - S_{\tau_{j-1}^k}$. Since $S - S_0 = (S \mathbb{1}_{\{S > 0\}}) \cdot R$, we can write (5.4) as
\begin{equation}
(5.5) \log \frac{X^{(x,\pi, \mathbb{T}^k)}}{X^{(x,\pi)}} = \sum_{j=1}^{k} \log \left( 1 + \eta_{j+1} \Delta_j^k \right) - \left( \eta \cdot S - \frac{1}{2}[\eta \cdot S^c, \eta \cdot S^c] - \sum_{t \leq \tau_k} (\eta_t \Delta S_t - \log (1 + \eta_t \Delta S_t)) \right).
\end{equation}
Since $(\mathbb{T}^k)_{k \in \mathbb{N}}$ converges to the identity and $\eta$ is càglàd, the dominated convergence theorem for stochastic integrals gives $\text{ucP}-\lim_{k \to \infty} \sum_{j=1}^{k} \eta_{j} \Delta_j^k = \eta \cdot S$. Further, using the fact that $x - \log(1 + x)$ behaves like $x^2/2$ near $x = 0$, standard stochastic-analysis manipulation shows that
\begin{align*}
\text{ucP}-\lim_{k \to \infty} \sum_{j=1}^{k} \left( \eta_{j+1} \Delta_j^k - \log \left( 1 + \eta_{j+1} \Delta_j^k \right) \right) &= \sum_{t \leq \tau_k} (\eta_t \Delta S_t - \log (1 + \eta_t \Delta S_t)) \\
&\quad + \frac{1}{2}[\eta \cdot S^c, \eta \cdot S^c].
\end{align*}
The last facts, coupled with (5.5), readily imply that $\text{ucP}-\lim_{k \to \infty} \log X^{(x,\pi, \mathbb{T}^k)} = \log X^{(x,\pi)}$, which completes the proof. \hfill \Box

\subsection*{5.4. Proof of Theorem 5.1 when $S$ satisfies Assumption 1.1.}
We state two helpful lemmata that, combined with Theorem 5.2, will prove Theorem 5.1.

\begin{lemma}
Let $R$ be a $d$-dimensional semimartingale with $\Delta R^i \geq -1$ for all $i = 1, \ldots, d$. For any $\Delta^d$-valued, predictable process $\pi$, there exists a sequence $(\pi^k)_{k \in \mathbb{N}}$ of $\Delta^d$-valued, predictable, simple (i.e., of buy-and-hold type) processes such that $\text{ucP}-\lim_{k \to \infty} \pi^k \cdot R = \pi \cdot R$ as well as $\mathbb{P}-\lim_{k \to \infty} [(\pi^k - \pi) \cdot R, (\pi^k - \pi) \cdot R]_T = 0$.
\end{lemma}

\begin{proof}
Assume without loss of generality that $\pi$ vanishes outside $[0,T]$ for some $T \in \mathbb{R}$. For any $\epsilon > 0$, one can find $v^1, \ldots, v^m$ in $\Delta^d$ and predictable sets $\Sigma^1, \ldots, \Sigma^m$ such that, with $\bar{\pi} := \sum_{i=1}^{m} v^i \mathbb{1}_{\Sigma^i}$, we have
\[ \mathbb{P} \left\{ \sup_{t \in [0,T]} \left| \int_0^t (\pi_t - \bar{\pi}_t) \, dR_t \right| > \epsilon, \text{ or } \left| (\pi_t - \bar{\pi}_t) \cdot R, (\pi_t - \bar{\pi}_t) \cdot R \right|_T > \epsilon \right\} < \epsilon. \]
From this approximation, it follows that we need only consider the case where \( \pi = v \| \Sigma \) where \( v \in \Delta^d \) and \( \Sigma \) is predictable.

The predictable \( \sigma \)-algebra on \( \Omega \times \mathbb{R}_+ \) is generated by the algebra of simple predictable sets of the form \( \bigcup_{j=1}^{n} H_{j-1} \times (t_{j-1}, t_j) \), where \( n \in \mathbb{N}, 0 = t_0 < \ldots < t_n \) and \( H_{j-1} \in \mathcal{F}_{t_{j-1}} \) for \( j = 1, \ldots, n \). A straightforward use of monotone class arguments shows that only the case where \( \Sigma \) is simple predictable needs to be dealt with, in which case the claim of the Lemma is obvious, since we are already dealing with a simple integrand. \( \square \)

Lemma 5.4. Consider a sequence \(( R^k )_{k \in \mathbb{N}} \) of semimartingales with \( \Delta R^k > -1 \) for all \( k \in \mathbb{N} \) such that \( \text{ucP-\textrm{lim}_{k \to \infty}} R^k = R \) as well as \( \text{ucP-\textrm{lim}_{k \to \infty}} [R^k, R^k] = [R, R] \) for some semimartingale \( R \) with \( \Delta R > -1 \). Then we also have \( \text{ucP-\textrm{lim}_{k \to \infty}} \mathcal{E}(R^k) = \mathcal{E}(R) \).

**Proof.** We have \( \mathcal{E}(R^k) > 0 \) for all \( k \in \mathbb{N} \) as well as \( \mathcal{E}(R) > 0 \). Then, the claim is obvious as long as one writes down

\[
\log \left( \frac{\mathcal{E}(R^k)}{\mathcal{E}(R)} \right) = R^k - R - \frac{1}{2} \left( [R^k, R^k]^c - [R, R]^c \right) - \sum_{t \leq \cdot} \left( \Delta R^k_t - \Delta R_t - \log \left( \frac{1 + \Delta R^k_t}{1 + \Delta R_t} \right) \right)
\]

and use \( \text{ucP-\textrm{lim}_{k \to \infty}} [R^k, R^k] = [R, R] \) and \( \text{ucP-\textrm{lim}_{k \to \infty}} R^k = R \), which also imply that \( \text{ucP-\textrm{lim}_{k \to \infty}} [R^k, R^k]^c = [R, R]^c \) and \( \text{ucP-\textrm{lim}_{n \to \infty}} \Delta R^n = \Delta R \). \( \square \)

Consider now \( X \equiv X^{(x, \pi)} \in \mathcal{X}_\Delta(x, T) \) for some \( \Delta^d \)-valued predictable process \( \pi \). To prove statement (2) of Theorem 5.1 we can safely assume that \( X \geq \epsilon \) for some \( \epsilon > 0 \), since if \( X \in \mathcal{X}_\Delta(x, T) \) then \( \epsilon + (1 - \epsilon/x)X \in \mathcal{X}_\Delta(x, T) \) as well. In this case, Lemmata 5.3 and 5.4 together provide us with a sequence of simple \( \Delta^d \)-valued predictable processes \((\pi^k)_{k \in \mathbb{N}} \) such that \( \text{ucP-\textrm{lim}_{k \to \infty}} X^{(x, \pi^k)} = X^{(x, \pi)} \). One can now invoke Theorem 5.2 and get a sequence \((X^k)_{k \in \mathbb{N}} \) of \( \mathcal{X}_\Delta(x, T) \)-valued processes with \( \text{ucP-\textrm{lim}_{k \to \infty}} X^k = X \), with concludes the proof of statement (2) of Theorem 5.1.

6. **Proofs of Results from Sections 2, 3 and 4**

6.1. **Proof of Theorem 2.3.** Since the proofs of (2) \( \Rightarrow \) (1) and (2) \( \Rightarrow \) (3) have been discussed, it suffices to prove (1) \( \Rightarrow \) (3), (3) \( \Rightarrow \) (1) and (3) \( \Rightarrow \) (2).

6.1.1. (1) \( \Rightarrow \) (3). We assume that \( S \) satisfies the assumptions of §2.4.4, which are weaker than the ones in Assumption 1.1. We also assume that condition NUPBR\( _\Delta \) is in force.

Start by defining the set of all dyadic rational numbers \( \mathbb{D} := \{ m/2^n \mid k \in \mathbb{N}, m \in \mathbb{N} \} \), which is dense in \( \mathbb{R}_+ \). Define also \( \mathbb{T}^k := \{ 0 < 1/2^k < \ldots < (k2^k - 1)/2^k < k \} \). We have \( \mathbb{T}^k \subset \mathbb{T}^{k'} \) for \( k < k' \) and \( \bigcup_{k \in \mathbb{N}} \mathbb{T}^k = \mathbb{D} \). In what follows below, we simplify notation using \( \mathcal{X}^n_\Delta \) to mean \( \mathcal{X}^{\Delta^n}_{\Delta^n} \) for the wealth-process classes.

Under condition NUPBR\( _\Delta \), one can find a numéraire portfolio in all classes \( \mathcal{X}^k_\Delta \), that is, a wealth process \( \tilde{X}^k \in \mathcal{X}^k_\Delta(1) \) such that \( X/\tilde{X}^k \) is a supermartingale for all \( X \in \mathcal{X}^k_\Delta(1) \), when sampled at times from \( \mathbb{T}^k \). In more detail, defining \( \tilde{Y}^k := 1/\tilde{X}^k \), we have \( \mathbb{E}[\tilde{Y}^k_t X_t \mid \mathcal{F}_s] \leq \tilde{Y}^k_s X_s \) for all \( X \in \mathcal{X}^k_\Delta(1), s < t \) and \( s, t \) times in \( \mathbb{T}^k \).
For all $k \in \mathbb{N}$, every $\tilde{Y}^k$ satisfies $\tilde{Y}_0^k = 1$ and is a positive supermartingale when appropriately sampled from times in $\mathbb{T}^k$; therefore, it is easily seen that for any $T \in \mathbb{D}$, the convex hull of the set $\{\tilde{Y}_T^k\}_{k \in \mathbb{N}}$ is bounded in probability. We also claim that, under NUPBR$_\Delta$, for any $T \in \mathbb{R}$, the convex hull of the set $\{\tilde{Y}_T^k\}_{k \in \mathbb{N}}$ is bounded away from zero in probability. Indeed, for any collection $(\alpha^k)_{k \in \mathbb{N}}$ such that $\alpha^k \geq 0$ for all $k \in \mathbb{N}$, having all but a finite number of $\alpha^k$’s is non-zero and satisfying $\sum_{k=1}^\infty \alpha^k = 1$, we have

$$\frac{1}{\sum_{k=1}^\infty \alpha^k Y^k} \geq \sum_{k=1}^\infty \alpha^k \frac{1}{Y^k} \geq \sum_{k=1}^\infty \alpha^k \tilde{X}^k \in \mathcal{X}_\Delta(1).$$

Under NUPBR$_\Delta$ the set $(X_t)_{X \in \mathcal{X}_\Delta(1,t)}$ is bounded in probability for all $t \in \mathbb{R}$, which proves that the convex hull of the set $\{\tilde{Y}_T^k\}_{k \in \mathbb{N}}$ is bounded away from zero in probability.

Using Lemma A.1 of [DS94], one proceeds as in the proof of Lemma 5.2(a) in [FK97] to infer the existence of a sequence $(\tilde{Y}_T^k)_{k \in \mathbb{N}}$ and some process $(\tilde{Y}_T)_{T \in \mathbb{D}}$ such that $\tilde{Y}^k$ is a convex combination of $\tilde{Y}_T^k, \tilde{Y}_T^{k+1}, \ldots$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} \tilde{Y}_T^k = Y_T$ for all $T \in \mathbb{D}$, $\mathbb{P}$-almost surely. The discussion of the preceding paragraph ensures that

$$(6.1) \quad \mathbb{P}[0 < Y_t < \infty, \forall t \in \mathbb{D}] = 1.$$

Pick any times $s < t$ in $\mathbb{D}$; we have $s$ and $t$ being elements of $\mathbb{T}^k$ for all $k$ large enough. The conditional version of Fatou’s Lemma will therefore give that, for all $X \in \bigcup_{k=1}^\infty \mathcal{X}^k$, $\mathbb{P}$-almost surely and exists in view of the supermartingale property of $\tilde{Y}$ when sampled from $\mathbb{D}$. It is easily seen that $Y$ is a càdlàg process and (6.1) gives $\mathbb{P}[0 < Y_t < \infty, \forall t \in \mathbb{R}_+] = 1$. Right-continuity of the filtration $\mathbb{F}$, coupled with (6.2), easily imply that $\mathbb{E}[Y_t X_t | \mathcal{F}_s] \leq \liminf_{k \to \infty} \mathbb{E}[\tilde{Y}_T^k X_t | \mathcal{F}_s] \leq \liminf_{k \to \infty} \tilde{Y}_T^k X_s = \tilde{Y}_s X_s$.

It follows that $(\tilde{Y}_T X_t)_{t \in \mathbb{D}}$ is a supermartingale (we look at the process $\tilde{Y} X$ only at times contained in $\mathbb{D}$) for all $X \in \bigcup_{k=1}^\infty \mathcal{X}^k$.

For any $t \in \mathbb{R}_+$ define $Y_t := \lim_{s \downarrow t, s \in \mathbb{D}} \tilde{Y}_s$ — the limit is $\mathbb{P}$-almost sure and exists in view of the supermartingale property of $\tilde{Y}$ when sampled from $\mathbb{D}$. It is easily seen that $Y$ is a càdlàg process and (6.1) gives $\mathbb{P}[0 < Y_t < \infty, \forall t \in \mathbb{R}_+] = 1$. Right-continuity of the filtration $\mathbb{F}$, coupled with (6.2), easily imply that $\mathbb{E}[Y_t X_t | \mathcal{F}_s] \leq Y_s X_s$ for all $s < t$ times in $\mathbb{R}_+$ and $X \in \bigcup_{k=1}^\infty \mathcal{X}^k$.

We have constructed essentially a supermartingale deflator for the class $\bigcup_{k=1}^\infty \mathcal{X}^k$. For the localizing sequence $(t_m)_{m \in \mathbb{N}}$ of §2.4.4 we have $m + S^m \in \mathcal{X}_\Delta^k$ for all $k \in \mathbb{N}$, where $S^m$ is the process $S$ stopped at $t_m$. It follows that $Y(m + S^m)$ is a supermartingale, thus a semimartingale. Since both $Y$ and $1/Y$ are semimartingales we have that $S^m$ is a semimartingale for all $m \in \mathbb{N}$, which finally gives that $S$ is a semimartingale.

6.1.2. $(3) \Rightarrow (1) \text{ and } (3) \Rightarrow (2)$. We discuss a strengthening of these two implications; we shall show that $(3)$ implies $\mathcal{Y}_C \neq \emptyset$ for some constraint set $\mathcal{C}$ satisfying the assumptions in §2.4.4, with $\mathcal{C}(\omega, t)$ containing the simplex for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. The fact that $\mathcal{Y}_C \subseteq \mathcal{Y}_\Delta$ will then imply condition (2) of Theorem 2.3, and of course then that (1) holds as well.

Since $S$ is a semimartingale, we can talk about continuous trading. Let $\mathcal{X}_C$ be the subset of elements $X^{x, \theta} \in \mathcal{X}$ such that $\theta \in X^{x, \theta} \mathcal{C}_-$, where $X^{x, \theta}$ and $\mathcal{C}_-$ denote the left-continuous versions of $X^{x, \theta}$ and $\mathcal{C}$ respectively. Both process $X^{x, \theta}$ and $\mathcal{C}_-$ are predictable and locally bounded — for the notion of predictability of set-valued process one can check
Appendix 1 of [KK07]. It follows that if $X^{x,\theta}$ is to be an element of $\mathcal{X}_\mathcal{E}$, $\theta$ must be a locally bounded process. In turn, this implies that there will exist a numéraire portfolio in $\mathcal{X}_\mathcal{E}$, i.e., some $\tilde{X} \in \mathcal{X}_\mathcal{E}(1)$ such that $X/\tilde{X}$ is a supermartingale for all $X \in \mathcal{X}_\mathcal{E}(1)$. This of course implies that $(1/\tilde{X}) \in \mathcal{Y}_\mathcal{E}$ and therefore that $\mathcal{Y}_\mathcal{E} \neq \emptyset$.

6.2. **Proof of Theorem 3.1.** The only implication that remains to be proved is $(1) \Rightarrow (3)$. Therefore, we assume that NUPBR holds.

The fact that $S$ must be a semimartingale is a consequence of Theorem 2.3. Now, in view of Theorem 5.1(1), we only need to show that if condition (3) fails, for some $T \in \mathbb{R}_+$ we have $(X_T)_{X \in \mathcal{X}(1,T)}$ being unbounded in probability.

Suppose that one cannot find a predictable $d$-dimensional process $\rho$ such that $A = \int_0^\infty (c_t \rho_t) dG_t$. In that case, linear algebra combined with a measurable selection argument give the existence of some $T \in \mathbb{R}_+$ and some bounded predictable process $\theta$ such that (a) $\int_0^T \theta_t dG_t = 0$; (b) $\int_0^T \langle \theta_t, dA_t \rangle$ is an increasing process for $t \in [0, T]$ and (c) $\mathbb{P}[\int_0^T \langle \theta_t, dA_t \rangle > 0] > 0$. This of course means that $X^{1,\theta} \in \mathcal{X}(1)$ satisfies $X^{1,\theta} \geq 1$, $\mathbb{P}[\{X^{1,\theta} \geq 1 \}] > 0$. Then, $X^{1,k\theta} \in \mathcal{X}(1)$ for all $k \in \mathbb{N}$ and $(X^{1,k\theta})_{k \in \mathbb{N}}$ is unbounded in probability.

Now we know that under NUPBR there exists a predictable $d$-dimensional process $\rho$ such that $A = \int_0^\infty (c_t \rho_t) dG_t$ — suppose that we had $\mathbb{P}[\int_0^T \langle \rho_t, c_t \rho_t \rangle dG_t = \infty] > 0$ for some $T \in \mathbb{R}_+$. Consider the sequence $\pi^k := \rho \mathbb{1}_{\{|\rho| \leq k\}}$ and let $X^k$ be defined via $X^k_0 = 1$ and satisfying $dX^k_t = X^k_t \frac{\pi^k_t}{\pi^k_t} dS_t$. Then, for all $k \in \mathbb{N}$, we have

$$\log X^k_T = -\frac{E^k_T}{2} + \int_0^T \langle \rho_t, c_t \rho_t \rangle dM_t,$$

where $E^k_T := \int_0^T \langle \rho_t, c_t \rho_t \rangle \mathbb{1}_{\{|\rho_t| \leq k\}} dG_t$ coincides with the total quadratic variation up to time $T$ of the local martingale $\int_0^T \langle \rho_t, c_t \rho_t \rangle dM_t$. It follows that for every $k \in \mathbb{N}$, one can find a one-dimensional standard Brownian motion $\beta^k$ such that

$$\log X^k_T = -\frac{E^k_T}{2} + \beta^k_{E^k_T}.$$

The strong law of large numbers for Brownian motion will imply that

$$\lim_{k \to \infty} \mathbb{P} \left[ \frac{\beta^k_{E^k_T}}{E^k_T} > \epsilon, \int_0^T \langle \rho_t, c_t \rho_t \rangle dG_t = \infty \right] = 0,$$

so that

$$\lim_{k \to \infty} \mathbb{P} \left[ \log X^k_T \frac{E^k_T}{E^k_T} > \frac{1}{2} - \epsilon, \int_0^T \langle \rho_t, c_t \rho_t \rangle dG_t = \infty \right] = 1,$$

Choosing $0 < \epsilon < 1/2$, we have that $\mathbb{P}[\int_0^T \langle \rho_t, c_t \rho_t \rangle dG_t = \infty] > 0$ implies that the sequence $(X^k_T)_{k \in \mathbb{N}}$ is unbounded in probability, which contradicts NUPBR.

6.3. **Proof of Theorem 4.1.** We give below in §6.3.1, §6.3.2 and §6.3.3 the proof of statements (1), (2) and (3) of Theorem 4.1, respectively. The proof of statement (4) follows the same lines, and is therefore omitted.
6.3.1. We begin by proving that $u_\Delta = \overline{\pi}_\Delta$. Assume first that $\overline{\pi}_\Delta$ is finite. Since

$$\lim_{\epsilon \downarrow 0} \overline{\pi}_\Delta(x - \epsilon) = \overline{\pi}_\Delta(x)$$

for all $x > 0$, it suffices to prove that for all $\epsilon \in (0, x)$ there exists an $\mathcal{X}_\Delta(x, T)$-valued sequence $(X^n)_{n \in \mathbb{N}}$ such that

$$\overline{\pi}_\Delta(x - \epsilon) \leq \liminf_{n \to \infty} \mathbb{E}[U(X^n_T)] + \epsilon.$$ 

Pick $\xi \in \mathcal{X}_\Delta(x - \epsilon, T)$ such that $\mathbb{E}[U(\xi_T)] \geq \overline{\pi}_\Delta(x - \epsilon)$; then, $X := \epsilon + \xi$ satisfies

$$\mathbb{E}[U(X_T)] \geq \overline{\pi}_\Delta(x - \epsilon), \quad X \in \mathcal{X}_\Delta(x, T)$$

and $X \geq \epsilon$. According to Theorem 5.1, we can find an $\mathcal{X}_\Delta(x, T)$-valued sequence $(X^n)_{n \in \mathbb{N}}$ with $\mathbb{P}$-$\lim_{n \to \infty} X^n_T = X_T$ and $X^n_T \geq \epsilon$. Fatou’s lemma implies that $\mathbb{E}[U(X_T)] \leq \liminf_{n \to \infty} \mathbb{E}[U(X^n_T)]$ and the proof that $u_\Delta = \overline{\pi}_\Delta$ for the case of finitely-valued $\pi_\Delta$ is clarified. The case where $\overline{\pi}_\Delta \equiv \infty$ is treated similarly.

6.3.2. We proceed now in showing that for any $\mathcal{X}_\Delta(x, T)$-valued sequence $(X^k)_{k \in \mathbb{N}}$ and any $\mathcal{X}_\Delta(x, T)$-valued sequence $(\overline{X}^k)_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} \mathbb{E}[U(X^k_T)] = u_\Delta(x) = \overline{\pi}_\Delta(x) = \lim_{k \to \infty} \mathbb{E}[U(\overline{X}^k_T)]$, we have $\mathbb{P}$-$\lim_{k \to \infty} |X^k_T - \overline{X}^k_T| = 0$.

For any $m \in \mathbb{N}$ define

$$(6.3) \quad K_m := \{(a, b) \in \mathbb{R}^2 \mid a \in [0, m], b \in [0, m] \text{ and } |a - b| > 1/m\}.$$ 

Under NUPBR, both $(X^k_T)_{k \in \mathbb{N}}$ and $(\overline{X}^k_T)_{k \in \mathbb{N}}$ are bounded in probability. Therefore, in order to prove that $\mathbb{P}$-$\lim_{k \to \infty} |X^k_T - \overline{X}^k_T| = 0$, we need to establish that, for all $m \in \mathbb{N}$,

$$\lim_{k \to \infty} \mathbb{P}\left[\left(X^k_T, \overline{X}^k_T\right) \in K_m\right] = 0.$$ 

Fix some $m \in \mathbb{N}$; the strict concavity of $U$ implies the existence of some $\beta_m > 0$ such that for all $(a, b) \in (0, \infty)^2$ we have

$$\frac{U(a) + U(b)}{2} + \beta_m I_{K_m}(a, b) \leq U\left(\frac{a + b}{2}\right),$$

for the set $K_m$ of (6.3).

Setting $a = X^k_T, b = \overline{X}^k_T$ in the previous inequality and taking expectations, one gets

$$\beta_m \mathbb{P}\left[\left(X^k_T, \overline{X}^k_T\right) \in K_m\right] \leq \mathbb{E}\left[U\left(\frac{X^k_T + \overline{X}^k_T}{2}\right)\right] - \frac{\mathbb{E}[U(X^k_T)] + \mathbb{E}[U(\overline{X}^k_T)]}{2}$$

$$\leq \overline{\pi}_\Delta(x) - \frac{\mathbb{E}[U(X^k_T)] + \mathbb{E}[U(\overline{X}^k_T)]}{2};$$

since $\lim_{k \to \infty} (\mathbb{E}[U(X^k_T)] + \mathbb{E}[U(\overline{X}^k_T)]) = 2\overline{\pi}_\Delta(x), \lim_{k \to \infty} \mathbb{P}\left[\left(X^k_T, \overline{X}^k_T\right) \in K_m\right] = 0$ follows.

6.3.3. Assume now all conditions of statement (3) in Theorem 4.1. Take any $\mathcal{X}_\Delta(x, T)$-valued sequence $(X^k)_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} \mathbb{E}[U(X^k_T)] = \overline{\pi}_\Delta(x)$. We already know from part (2) of Theorem 4.1 that $\mathbb{P}$-$\lim_{k \to \infty} X^k_T = \overline{X}_T$. What remains now is to pass to the stronger convergence $uc\mathbb{P}$-$\lim_{k \to \infty} X^k = \overline{X}$. Observe that since $\inf_{t \in [0, T]} \overline{X}_t > 0$ (this is a consequence of $\overline{X} > 0$ and the NUPBR property), the latter convergence is equivalent to $uc\mathbb{P}$-$\lim_{k \to \infty} (X^k/\overline{X}) = 1$. Now, $\overline{X}$ is a maximal element in $\mathcal{X}_\Delta(x, T)$, meaning that for any other $\xi \in \mathcal{X}_\Delta(x, T)$ with $\mathbb{P}[\xi_T \geq \overline{X}_T] = 1$ we actually have $\mathbb{P}[\xi_T = \overline{X}_T] = 1$. This means that there exists a probability $Q \sim \mathbb{P}$ such that $X/\overline{X}$ is a Q supermartingale for all $X \in \mathcal{X}_\Delta(x, T)$ — for this, see for example [DS95]. Letting $Z^k := X^k/\overline{X}$ for all $k \in \mathbb{N}$, we are in the following situation: $Z^k$ a Q-supermartingale with $Z^k_0 = 1$ for all $k \in \mathbb{N}, Z^k_t = Z^k_T$
for all \( t > T \) and \( \mathbb{Q}\)-lim_{k→∞} Z^k_T = 1; the next proposition shows that \( \text{ucQ}\)-lim_{k→∞} Z^k = 1 and completes the proof of Theorem 4.1.

**Proposition 6.1.** On a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})\), let \((Z^k)_{k \in \mathbb{N}}\) be a sequence of nonnegative supermartingales with \( Z^k_0 = 1 \) such that \( \mathbb{Q}\)-lim_{k→∞} Z^k_T = 1 \(, \) where \( T \) is a finite \( \mathbb{F}\)-stopping time. We then have \( \mathbb{Q}\)-lim_{k→∞} sup_{t∈[0,T]} |Z^k_t| = 0.

**Proof.** Since all \( Z^k \), \( k \in \mathbb{N} \) are positive, it suffices to show that \( \mathbb{Q}\)-lim_{k→∞} sup_{t∈[0,T]} Z^k_t = 1 \) and \( \mathbb{Q}\)-lim_{k→∞} inf_{t∈[0,T]} Z^k_t = 1. We tackle these two claims in the next paragraphs.

As a warm-up for proving \( \mathbb{Q}\)-lim_{k→∞} sup_{t∈[0,T]} Z^k_t = 1 \), observe that \( \text{lim}_{k→∞} \mathbb{E}[Z^k_T] = 1 \) as a consequence of Fatou’s lemma; this implies the \( \mathbb{Q}\)-uniform integrability of \((Z^k)_{k \in \mathbb{N}}\) and thus we obtain \( \lim_{k→∞} \mathbb{E}[|Z^k_T| - 1] = 0 \). In particular, the probabilities \((\mathbb{Q}^k)_{k \in \mathbb{N}}\) defined on \((\Omega, \mathcal{F}_T)\) via \((d\mathbb{Q}^k/d\mathbb{Q})|_{\mathcal{F}_T} = Z^k_T/\mathbb{E}[Z^k_T] \) are all equivalent, and converge in total variation to \( \mathbb{Q} \).

Fix \( \epsilon > 0 \) and let \( \tau^k := \inf\{t \in [0,T] \mid Z^k_t > 1+\epsilon\} \wedge T \). We have \( \mathbb{E}[Z^k_T] = \mathbb{E}[Z^k_{\tau^k}] \leq 1 \), which means that \( \text{lim}_{k→∞} \mathbb{E}[Z^k_{\tau^k}] = 1 \). Showing that \( \lim_{k→∞} \mathbb{P}[\tau^k < T] = 0 \) will imply (since \( \epsilon > 0 \) is arbitrary) that \( \mathbb{Q}\)-lim_{k→∞} sup_{t∈[0,T]} Z^k_t = 1 \). Suppose on the contrary (passing to a subsequence if necessary) that \( \lim_{k→∞} \mathbb{Q}[\tau^k < T] = p > 0 \). Then,

\[
1 = \text{lim}_{k→∞} \mathbb{E}[Z^k_{\tau^k}] \geq (1 + \epsilon)p + \text{lim}_{k→∞} \mathbb{E}[Z^k_T \mathbb{1}_{\{\tau^k = T\}}] = (1 + \epsilon)p + \text{lim}_{k→∞} \left( \mathbb{E}[Z^k_T] \mathbb{Q}^{k}[\tau^k = T]\right) = 1 + \epsilon p,
\]

where the last equality follows from \( \text{lim}_{k→∞} \mathbb{E}[Z^k_T] = 1 \) and \( \text{lim}_{k→∞} \mathbb{Q}^{k}[\tau^k = T] = \mathbb{Q}[\tau^k = T] = 1 - p \). This contradicts the fact that \( p > 0 \) and the first claim is proved.

Again, with fixed \( \epsilon > 0 \), redefine \( \tau^k := \inf\{t \in [0,T] \mid Z^k_t < 1-\epsilon\} \wedge T \) — we only need to show that \( \lim_{k→∞} \mathbb{P}[\tau^k < T] = 0 \). Observe that on the event \( \{\tau^k < T\} \) we have \( \mathbb{Q}[Z^k_T > 1-\epsilon^2 \mid \mathcal{F}_{\tau^k}] \leq (1-\epsilon)/(1-\epsilon^2) = 1/(1+\epsilon) \). Then,

\[
\mathbb{Q}[Z^k_T > 1-\epsilon^2] = \mathbb{E}[\mathbb{Q}[Z^k_T > 1-\epsilon^2 \mid \mathcal{F}_{\tau^k}]] \leq \mathbb{Q}[\tau^k = T] + \mathbb{Q}[\tau^k < T] \frac{1}{1+\epsilon},
\]

Rearranging and taking the limit as \( k \) goes to infinity we get

\[
\text{lim sup}_{k→∞} \mathbb{Q}[\tau^k < T] \leq \frac{1+\epsilon}{\epsilon} \text{lim sup}_{k→∞} \mathbb{Q}[Z^k_T \leq 1-\epsilon^2] = 0,
\]

and completes the proof of the Proposition. \( \square \)

**References**


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