Law of the Minimal Price

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Abstract. This paper introduces a realistic, generalized market modeling framework for which the Law of One Price no longer holds. Instead the Law of the Minimal Price will be derived, which for contingent claims with long term to maturity may provide significantly lower prices than suggested under the currently prevailing approach. This new law only requires the existence of the numeraire portfolio, which turns out to be the portfolio that maximizes expected logarithmic utility. In several ways it will be shown that the numeraire portfolio cannot be outperformed by any nonnegative portfolio. The new Law of the Minimal Price leads directly to the real world pricing formula, which uses the numeraire portfolio as numeraire and the real world probability for calculating conditional expectations. The cost efficient pricing and hedging of extreme maturity zero coupon bonds illustrates the new law in the context of the US market.

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1 Introduction

Classical asset pricing theories, as developed in Debreu (1959), Sharpe (1964), Lintner (1965), Merton (1973a, 1973b), Ross (1976) and Harrison & Kreps (1979), yield the Law of One Price, where a replicable payoff can only be hedged by portfolios that all have the same unique price. A description of various classical pricing approaches can be found, for instance, in Cochrane (2001). The current paper proposes a more general and likely more realistic modeling framework, the benchmark approach. This framework can accommodate almost any reasonable market model. However, it no longer accommodates the Law of One Price. It only requires the existence of a benchmark, the numeraire portfolio, which was originally introduced by Long (1990) in a rather special setting. Nonnegative portfolios, when denominated in units of this portfolio, trend downward or are at most trendless, as shown in Becherer (2001), Bühlmann & Platen (2003), Platen & Heath (2006) and Karatzas & Kardaras (2007). The numeraire portfolio can be used as benchmark in investment management and as numeraire in derivative pricing. In a very general setting it will turn out to equal the growth optimal portfolio, which maximizes expected logarithmic utility from terminal wealth and was originally discovered in Kelly (1956).

For jump-diffusion markets the benchmark approach has been described in Platen & Heath (2006). The current paper extends this approach more generally and reveals surprising properties of some securities with important practical implications. Only a few basic statistical arguments will be required to establish the fundamental properties of general financial market models under the new approach.

The most striking feature of the richer modeling framework is the possible co-existence of several self-financing portfolios that perfectly replicate the same payoff but exhibit totally different prices. The presence of such different replicating portfolios contradicts the classical Law of One Price. Instead, the Law of the Minimal Price will be derived. It identifies for a given contingent claim the minimal replicating portfolio process as the economically correct minimal possible price.

By exploiting this new law the paper will illustrate for the US market how to replicate in two different ways a fixed cash amount at a given maturity date. One method purely invests in the savings account following a, so called, savings bond. The other prices and hedges the less expensive fair zero coupon bond, which represents the minimal possible replicating portfolio. In Figure 1.1, the logarithm of both self-financing portfolio processes are displayed over the period from 1920 until 2007, with details given later. This figure shows also the logarithm of the savings account that starts at the initial value of the zero coupon bond. It will be demonstrated that the zero coupon bond systematically outperforms the savings account at maturity.
To formulate the new Law of the Minimal Price conveniently, call a security, when expressed in units of the numeraire portfolio, a benchmarked security. Furthermore, call a price process fair, when its current benchmarked value is the best forecast of its future benchmarked values. The Law of the Minimal Price identifies, for a replicable contingent claim, simply the corresponding fair price process. Under the benchmark approach pricing becomes an investment decision, whereas classical no-arbitrage arguments search only for a price that is consistent relative to other prices. This latter method does not always provide a unique solution. It will be shown that the co-existence of several replicating price processes for the same contingent claim illustrated in Figure 1.1, does not constitute strong arbitrage in a sense to be specified later. However, weaker forms of arbitrage may exist. Pricing by traditional no-arbitrage arguments may not lead to the least expensive price process that the Law of the Minimal Price identifies. This fact has severe consequences. In particular, medium and long term contingent claims, as typically used by pension funds, may be replicated with lower costs than suggested by traditional approaches.

The Law of the Minimal Price yields directly the real world pricing formula, where the expectation is taken with respect to the real world probability and the numeraire portfolio is the numeraire. No change of probability measure is necessary. The real world pricing formula generalizes the widely used risk neutral pricing formula, and also the traditional actuarial pricing formula. Both formulae represent the central pricing rules in their respective streams of literature.
2 A Two Asset Market

Before presenting a general theory the paper illustrates key features of the benchmark approach in a discrete time two-asset market. The investment universe consists of the strictly positive savings account \( S^0_t \) and a strictly positive risky primary security account \( S^1_t \). The latter may be interpreted as a strictly positive, diversified stock accumulation index, at the trading time, \( t = ih, i \in \{0, 1, \ldots\} \), \( h > 0 \). The asset ratio
\[
A^1_{t_i,h} = \frac{S^1_{t_i} + h}{S^0_{t_i}}
\]
of the index, which is simply the gross return plus one, can reach any finite strictly positive value and may follow a general stochastic process. The asset ratio
\[
A^0_{t_i,h} = \frac{S^0_{t_i} + h}{S^0_{t_i}}
\]
of the savings account \( S^0_t \) is assumed to be deterministic and always strictly positive, \( i \in \{0, 1, \ldots\} \). The central building block of the benchmark approach is the numeraire portfolio which needs to remain strictly positive. To ensure strict positivity of a portfolio \( S^\delta_t \), its fraction \( \pi^0_{\delta,t_i} \) invested in the savings account \( S^0_t \) has to stay in the interval \([0, 1]\).

In Section 5 the numeraire portfolio will be shown to be growth optimal. Therefore, it can be identified for the two asset market by maximizing the expected growth at time \( t_i \) for strictly positive portfolios \( S^\delta \), with \( S^\delta_0 = x > 0 \), given by the expression
\[
g^\delta_{t_i,h} = E_{t_i} \left( \ln \left( A^\delta_{t_i,h} \right) \right),
\]
with portfolio ratio
\[
A^\delta_{t_i,h} = \frac{S^\delta_{t_i} + h}{S^0_{t_i}} = \pi^0_{\delta,t_i} A^0_{t_i,h} + (1 - \pi^0_{\delta,t_i}) A^1_{t_i,h}.
\]

In relation (2.1) \( E_{t_i}(\cdot) \) denotes conditional expectation under the information available at time \( t_i \). We interpret \( \ln(y) \) as \(-\infty\) for \( y \leq 0 \). Assuming finite \( g^\delta_{t_i,h} \), its first derivative with respect to \( \pi^0_{\delta,t_i} \) equals
\[
\frac{\partial g^\delta_{t_i}}{\partial \pi^0_{\delta,t_i}} = \frac{E_{t_i} \left( Q_{t_i,h} \right)}{\pi^0_{\delta,t_i} Q_{t_i,h} + 1},
\]
with \( Q_{t_i,h} = \frac{A^0_{t_i,h}}{A^1_{t_i,h}} - 1 \). Since its second derivative turns out to be negative, the expected growth \( g^\delta_{t_i,h} \) has at most one genuine maximum with respect to \( \pi^0_{\delta,t_i} \). This maximum is characterized by the fraction \( \pi^0_{\delta,t_i} \), which satisfies the first order condition
\[
\frac{\partial g^\delta_{t_i,h}}{\partial \pi^0_{\delta,t_i}} = 0.
\]

Three cases can arise: \( \pi^0_{\delta,t_i} \in [0, 1], \pi^0_{\delta,t_i} < 0 \) or \( \pi^0_{\delta,t_i} > 0 \). First consider the case when the fraction \( \pi^0_{\delta,t_i} \) for the genuine maximum falls in the interval \([0, 1]\). This yields by (2.3) and (2.4) the equation
\[
E_{t_i} \left( \frac{A^1_{t_i,h}}{A^\delta_{t_i,h}} \right) = E_{t_i} \left( \frac{A^0_{t_i,h}}{A^\delta_{t_i,h}} \right),
\]
with portfolio ratio \( A_{t,i,h}^δ = \pi_{\delta,t_i}^0 A_{t,i,h}^0 + (1 - \pi_{\delta,t_i}^0) A_{t,i,h}^1 \), see (2.2). In this case one has a genuine maximum for the expected growth \( g^\delta_{t_i,h} \), and both securities \( S^0 \) and \( S^1 \) are constituents of the numeraire portfolio \( S_{t_i}^\delta = S_{t_i}^\delta \). Furthermore, in this case both primary security accounts turn out to be fair with their current benchmarked values providing the best forecast of their future benchmarked values. Consequently, all portfolios \( S^\delta \) in this case are fair, which can be seen from the equation

\[
E_{t_i} \left( \frac{S_{t_i+1}^\delta}{S_{t_i}^\delta} \right) = E_{t_i} \left( \frac{A_{t_i,h}^\delta}{A_{t_i,h}^\delta} \right) = E_{t_i} \left( \pi_{\delta,t_i}^0 \left( \frac{A_{t_i,h}^0}{A_{t_i,h}^\delta} \right) + (1 - \pi_{\delta,t_i}^0) \left( \frac{A_{t_i,h}^1}{A_{t_i,h}^\delta} \right) \right) = 1.
\]

The above discussed special case refers to a discrete time market where the Law of One Price holds because all portfolios are fair.

Consider now the remaining cases. For this purpose one can rewrite equation (2.3) in the form

\[
\frac{\partial g^\delta_{t_i,h}}{\partial \pi_{\delta,t_i}} = E_{t_i} \left( Q_{t_i,h} - \pi_{\delta,t_i}^0 \frac{(Q_{t_i,h})^2}{1 + \pi_{\delta,t_i}^0 Q_{t_i,h}} \right) \tag{2.7}
\]

with \( Q_{t_i,h} \) as in (2.3). Setting the partial derivative (2.7) to zero, characterizes by (2.4) the fraction \( \pi_{\delta,t_i}^0 \) for the genuine maximum of the expected growth \( g^\delta_{t_i,h} \). For vanishing time step size \( h = t_{i+1} - t_i \to 0 \) one obtains asymptotically from (2.7) and (2.4) the relation

\[
\pi_{\delta,t_i}^0 = \lim_{h \to 0} \frac{\frac{1}{h} E_{t_i+1}(Q_{t_i,h})}{\lim_{h \to 0} \frac{1}{h} E_{t_i+1} \left( \frac{(Q_{t_i,h})^2}{1 + \pi_{\delta,t_i}^0 Q_{t_i,h}} \right)} \tag{2.8}
\]

This identifies the fraction \( \pi_{\delta,t_i}^0 \) of the genuine maximum as the ratio of the expected return over the squared volatility of \( \frac{S_{t_i}^0}{S_{t_i}^\delta} \), which covers the fractions for the growth optimal portfolio in the Merton problem, see Merton (1973a). When \( \pi_{\delta,t_i}^0 \in [0, 1] \) one obtains the already discussed case.

In the second case, with \( \pi_{\delta,t_i}^0 < 0 \), the savings account is not fair since the optimal fraction \( \pi_{\delta,t_i}^0 \) of the numeraire portfolio \( S_{t_i}^\delta \) results as a corner solution under the strict positivity constraint for \( S_{t_i}^\delta \). In this case the index \( S^1 \) emerges as the numeraire portfolio. In the third case, when \( \pi_{\delta,t_i}^0 > 1 \), one obtains another corner solution with \( \pi_{\delta,t_i}^0 = 1 \). In this case the index is not fair and the savings account becomes the numeraire portfolio.

The US market appears to corresponds with the second case. To see an indication of this, one can use as short rate for the savings account of a generic investor the US 90 Day T Bill Rate plus 0.4%. As risky security one can employ the US stock accumulation index formed by Global Financial Data, based on monthly
S&P500 total returns. By using the index as numeraire portfolio the resulting benchmarked US savings account is shown in Figure 2.2 for the period from January 1920 until September 2007, which is in the following also called the benchmarked savings bond, since it is normalized such that it pays $1 at the end of the period. The benchmarked savings account shows a strong downward trend, which is the key feature that creates new effects under the general benchmark approach. This negative trend makes economic sense, since it reflects the presence of the well-known equity risk premium.

By using the observed \( n = 1052 \) annualized monthly returns of the benchmarked savings account, one observes a mean of about \(-3.96\%\) and a volatility of approximately \(19.0\%\). This is roughly consistent with what most studies found in varying settings, see, for instance, Mehra & Prescott (1985). The observed negative mean makes it likely that the US market fits the second case. The US stock accumulation index performed so well that under the constraint of a strictly positive numeraire portfolio, holding this index becomes growth optimal. The index in this two asset market should, therefore, be chosen as the numeraire portfolio.

For simplicity, it is assumed that the short rate is deterministic. By making it random one would only complicate the exposition, and would obtain very similar and even slightly more dramatic results, due to an effect on bond prices resulting from Jensen’s inequality. One can invest at time \( t \in [0, T] \), the amount \( P^*(t, T) = \frac{S_t}{S_T} \) in the savings account to replicate at the maturity date \( T > 0 \), the payoff of $1. Here \( P^*(t, T) \) denotes the price at time \( t \) of the savings bond with maturity \( T \). Its logarithm is shown in Figure 1.1, whereas its benchmarked value is displayed in Figure 2.1.

![Benchmarked savings bond and benchmarked fair zero coupon bond](image.png)

Figure 2.1: Benchmarked savings bond and benchmarked fair zero coupon bond.
The fair zero coupon bond price

\[ P(t, T) = S_t^\delta E_t \left( \frac{1}{S_T^\delta} \right) \]  

(2.9)

results from the following conditional expectation

\[ \frac{P(t, T)}{S_t^\delta} = E_t \left( \frac{1}{S_T^\delta} \right), \]

which will be explained later. For the moment note that it is simply chosen to be fair.

To quantify for the above two asset market the price of a fair zero coupon bond one needs to employ some model. The stylized version of the Minimal Market Model (MMM), see Platen & Heath (2006), is now employed, where at time \( t \) the expected trend of the discounted numeraire portfolio \( S_t^\delta = \frac{S_t^\delta}{S_0^\delta} \) is modeled by the exponential function \( \frac{\alpha}{\eta} \exp\{\eta t\} \). Under this two parameter model the benchmarked savings account is not fair, which is the case that needs to be modeled for the US market. Fitting the logarithm of the discounted index, for the period after the second world war from January 1945 until September 2007, by standard linear regression yields an estimate for the net growth rate \( \eta \) of about 0.0511, with an \( R^2 \) of 0.88, see Figure 2.2. The scaling parameter \( \alpha \) can also be estimated by linear regression. This regression exploits the fact that, under the stylized MMM, the slope of the approximate quadratic variation \( V_t = \sum_{\ell=1}^i (Z_{t_\ell} - Z_{t_{\ell-1}})^2, \ i \in \{1, 2, \ldots\}, \) of the square root of the normalized index \( Z_{t_\ell} = \sqrt{\frac{S_{t_\ell}^\delta}{\alpha \exp\{\eta t_\ell\}}} \) is approximately 0.25, for small \( h \), see Platen & Heath (2006). The linear regression of \( V_t \) in Figure 2.3 produces a large \( R^2 \) value of 0.995 for the
estimate $\alpha \approx 0.01429$. Under the stylized MMM the explicitly known transition density of the discounted numeraire portfolio $\tilde{S}^{\delta^*}$ yields the explicit formula

$$P(t, T) = P^*(t, T) \left(1 - \exp \left\{ - \frac{2 \eta \tilde{S}^{\delta^*}}{\alpha (\exp\{\eta T\} - \exp\{\eta t\})} \right\} \right)$$  \hspace{1cm} (2.10)$$

for the fair zero coupon bond price by (2.9) for $0 \leq t \leq T < \infty$, see Platen & Heath (2006). Figure 2.1 plots also the evolution of the benchmarked fair zero coupon bond price with maturity $T$ in September 2007. The price of the benchmarked fair zero coupon bond is initially significantly below that of the benchmarked savings bond. Only the benchmarked fair zero coupon bond attempts to provide with its current value the best forecast of the future benchmarked payoff. As will be explained later, it identifies in this way the minimal price process that replicates the given payoff at maturity.

Figure 2.4 displays the price evolution, in US dollar denomination, for the savings bond and the fair zero coupon bond. Both self-financing portfolios replicate the payoff of $1$ at maturity. However, they start with significantly different prices, which contradicts the classical Law of One Price. The savings bond has in January 1920 a price of $P^*(0, T) \approx 0.0255$. The fair zero coupon bond is far less expensive and priced at only $P(0, T) \approx 0.0008$. This fair price represents less than 3.2% of the price of the savings bond. One may argue that this result is potentially a feature of the stylized MMM. However, a look at Figure 2.1 shows that any reasonable model that captures well the distribution of the on average small benchmarked payoff $\frac{1}{S_T}$, will yield a corresponding small benchmarked zero coupon bond price.

For illustration of the phenomenon of systematic outperformance, which will be discussed later in Definition 4.5, Figure 2.4 also includes the savings account that starts at the price of the fair zero coupon bond. It reaches at the end of the
period only a value of about $0.031$ compared with the payoff $\$1.0$ of the fair zero coupon bond. The logarithms of these price processes were shown in Figure 1.1.

Finally, under the stylized MMM a self-financing hedge portfolio is formed for the fair zero coupon bond by holding at the times $t_i$ the corresponding number of units

$$
\delta^*_i = \frac{\partial P(t_i, T)}{\partial S^*_i}
$$

$$
= P^*(t_i, T) \exp \left\{ \frac{-2 \eta \delta^*_i}{\alpha (\exp{\eta T} - \exp{\eta t_i})} \right\} \frac{2 \eta}{\alpha (\exp{\eta T} - \exp{\eta t_i})}
$$

in the index and the remainder in the savings account. Figure 2.5 displays the
resulting fraction \( \frac{\delta^*_i S^0_t}{P(t_i, T)} \) to be invested at time \( t_i \) in the savings account. Initially almost no wealth is invested in the savings account. For a long time the fair zero coupon bond simply exploits the superior growth of the index. Closer to maturity the capital is then systematically shifted over to the savings account. In the hedge simulation, the self-financing hedge portfolio is started in January 1920 and during each month the number of units invested in the index is specified by the above hedge ratio. The benchmarked profit and loss is defined as the difference between the benchmarked fair zero coupon bond price and its initial benchmarked price, minus the benchmarked gains from trade. In the example the observed maximum absolute benchmarked profit and loss amounts only to about 0.00008. Consequently, if the hedge portfolio were also plotted in Figure 2.4 it would appear visually identical to the fair zero coupon bond. Similar hedge simulations with small hedge errors have been performed for European put and call options, as well as for digital options on the index, see Hulley & Platen (2008).

The above example demonstrates that the classical Law of One Price may be violated in reality. This observation opens new perspectives for the pricing and hedging of medium and extreme maturity derivatives. To explore these systematically, the benchmark approach can provide an appropriate theoretical backing. In what follows, this new approach is derived in more generality than is described in Platen & Heath (2006).

3 Financial Market

Consider a financial market in continuous time with \( d \) risky, nonnegative, primary securities, \( d \in \{1, 2, \ldots \} \). These could be, for instance, shares or currencies. Denote by \( S^j_t \) the value of the corresponding \( j \)th primary security account, \( j \in \{0, 1, \ldots , d\} \), at time \( t \geq 0 \). This nonnegative account holds units of the \( j \)th primary security plus its accumulated dividends or interest payments. The 0th primary security account \( S^0_t \) denotes the value of the savings account at time \( t \geq 0 \). The \( j \)th primary security account is characterized by its asset ratio

\[
A^j_{t,h} = \frac{S^j_{t+h}}{S^j_t}
\]

over the period \( [t, t + h] \), for \( S^j_t > 0; t, h \in [0, \infty); h > 0 \) and \( j \in \{0, 1, \ldots , d\} \).

The market participants can form self-financing portfolios with primary security accounts as constituents by using buy and hold strategies. At any possibly random but observable time, wealth can be reallocated. A portfolio value \( S^\delta_t \) at time \( t \) is described by the number \( \delta^j_t \) of units held in the \( j \)th primary security account \( S^j_t \) for all \( j \in \{0, 1, \ldots , d\}, t \geq 0 \). For simplicity, assume that the units of the primary security accounts are perfectly divisible, and that for all \( t \in [0, \infty) \) the values \( \delta^0_t, \delta^1_t, \ldots , \delta^d_t \), for any given strategy \( \delta = (\delta^0_t, \delta^1_t, \ldots , \delta^d_t)^\top, t \geq 0 \),
depend only on information available at time \( t \). At that time the portfolio value is given by

\[
S_t^\delta = \sum_{j=0}^{d} \delta_t^j S_t^j. \tag{3.2}
\]

Note that changes in the value of a self-financing portfolio are only due to changes in values of the primary security accounts. We only consider self-financing portfolios and assume no frictions. By \( \mathcal{V}_x^+ \) denote the set of all strictly positive self-financing portfolios, with initial capital \( x > 0 \).

### 4 Numeraire Portfolio

The benchmark approach employs a very special portfolio \( S^{\delta*} \in \mathcal{V}_x^+ \) as benchmark. It is in several ways the “best” performing strictly positive portfolio, as will be shown later.

**Definition 4.1** For given \( x > 0 \), a strictly positive finite portfolio \( S^{\delta*} \in \mathcal{V}_x^+ \) is called a numeraire portfolio, if for all observable times \( t \) and positive real numbers \( h > 0 \), the expected returns of all nonnegative portfolios \( S^\delta \), when denominated in units of \( S^{\delta*} \), are never greater than zero as long as \( S_t^\delta > 0 \), that is,

\[
E_t \left( \frac{S_t^{\delta+h}}{S_t^{\delta*}} \frac{S_t^\delta}{S_t^{\delta*}} - 1 \right) \leq 0. \tag{4.1}
\]

The notion of a numeraire portfolio was originally introduced by Long (1990) and later generalized in Bajeux-Besnainou & Portait (1997) and Becherer (2001). These authors worked under assumptions that would not cover the above MMM. More recently, Platen (2002), Bühlmann & Platen (2003), Platen & Heath (2006), Platen (2006) and Karatzas & Kardaras (2007) emphasized that in a more general setting, as long as a numeraire portfolio exists, one still obtains a viable financial market model. The benchmark approach makes the following extremely weak key assumption, which is satisfied for almost all models of practical interest, see Platen & Heath (2006) and Karatzas & Kardaras (2007).

**Assumption 4.2** For given \( x > 0 \), there exists a numeraire portfolio \( S^{\delta*} \in \mathcal{V}_x^+ \).

At time \( t \) the benchmarked value \( \hat{S}_t^\delta \) of a portfolio \( S^\delta \) is given by the ratio

\[
\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta*}} \tag{4.2}
\]
for all $t \geq 0$. One justification for the claim that the numeraire portfolio is the “best” performing portfolio was already given in Definition 4.1. It simply says that a numeraire portfolio performs so well that the expected return of any benchmarked nonnegative portfolio can never exceed zero. Consequently, for any benchmarked nonnegative portfolio $S^\delta$ the expected portfolio ratio never exceeds one, that is,
\[ E_t \left( \frac{S^\delta_{t+h}}{S^\delta_t} \right) \leq 1 \quad (4.3) \]
for all $t, h \geq 0$. Relation (4.1) leads directly to the following conclusion.

**Corollary 4.3** The benchmarked values of any nonnegative portfolio $S^\delta$ satisfy the inequality
\[ \hat{S}^\delta_t \geq E_t \left( \hat{S}^\delta_s \right) \quad (4.4) \]
for all $0 \leq t \leq s < \infty$.

Consequently, the currently observed benchmarked value of a nonnegative portfolio is always greater than or equal to its expected future benchmarked value at any future time. This means that if there were any trend in a benchmarked nonnegative portfolio, then this trend could only point downwards. Stochastic processes with this property are called **supermartingales**, see Shiryaev (1984). One can call relation (4.4) the supermartingale property. It is the central property of a financial market, as will become clearer below. This property opens, for instance, the powerful tool box of stochastic calculus of financial analysis, since all supermartingales obey this calculus.

Consider two strictly positive portfolios that are supposed to be numeraire portfolios. According to Corollary 4.3 the first portfolio, when expressed in units of the second, must satisfy the supermartingale property. By the same argument, the second portfolio, when expressed in units of the first, must also satisfy the supermartingale property. By Jensen’s inequality the portfolios must be identical. Consequently, for given initial capital $x > 0$, the value process $S^\delta \in \mathcal{V}_x^+$ of a numeraire portfolio is unique. Note that the stated uniqueness of the numeraire portfolio $S^\delta_\ast$ does not imply that the units invested in primary security accounts have to be unique, in particular, when redundant securities exist.

To demonstrate another manifestation of “best” performance of the numeraire portfolio $S^\delta_\ast$, define the long term growth rate $g^\delta$ of a strictly positive portfolio $S^\delta \in \mathcal{V}_x^+$ as the upper limit
\[ g^\delta = \limsup_{t \to \infty} \frac{1}{t} \ln \left( \frac{S^\delta_t}{S^\delta_0} \right). \quad (4.5) \]
This paper interprets limits and relations in the almost sure sense, see Shiryaev (1984). The long term growth rate (4.5) is defined pathwise and does not involve any expectation. By exploiting the supermartingale property, the following fascinating feature of the numeraire portfolio will be derived in the Appendix.
Theorem 4.4  The numeraire portfolio $S^δ∗ \in \mathcal{V}_x^+$ achieves the maximum long term growth rate. This means, when compared with any other strictly positive portfolio $S^δ \in \mathcal{V}_x^+$, one has

$$g^δ \leq g^δ∗.$$  \hspace{1cm} (4.6)

Consequently, the trajectory of the numeraire portfolio outperforms in the long run that of every other strictly positive portfolio which starts with the same initial capital. This property is independent of the model, as are all major properties we present. It justifies the choice of the numeraire portfolio as benchmark in fund management. We will see later that it is also the natural numeraire for pricing. An investor, who is aiming for the highest possible portfolio value in the long run, has to invest her or his total tradable wealth into the numeraire portfolio. Of course, the nature of the underlying market dynamics determines how long it takes for this long run behaviour to become apparent.

It is a challenge to identify the numeraire portfolio in practice. For a sequence of jump-diffusion markets with an increasing number of primary security accounts, the numeraire portfolio has been identified as the limit of diversified portfolios, see Platen & Heath (2006). Therefore, a diversified stock accumulation index, as the S&P500 total return index for the US market, can be interpreted as a proxy for the numeraire portfolio.

Over short and medium time periods, almost any strictly positive portfolio can generate larger returns than those exhibited by the numeraire portfolio. However, as will now be shown, such outperformance cannot be achieved systematically.

Definition 4.5  A nonnegative portfolio $S^δ$ systematically outperforms a strictly positive portfolio $S^δ$ if

(i) both portfolios start from equal initial capital $S^δ_{t_0} = S^δ_{t_0}$;

(ii) at a later time $t$, $S^δ_t$ is at least equal to $S^δ_t$, that is $P(S^δ_t \geq S^δ_t) = 1$ and

(iii) the probability for $S^δ_t$ being strictly greater than $S^δ_t$ is strictly positive, that is $P\left(S^δ_t > S^δ_t\right) > 0$.

Figure 1.1 provides an example, where a fair zero coupon bond systematically outperforms a savings account, when both start with the same initial capital. The above notion of systematic outperformance was introduced in Platen (2004). It also relates to the notion of relative arbitrage studied in Fernholz & Karatzas (2005) and the notion of a maximal element in Delbaen & Schachermayer (1998). Based on the supermartingale property we prove the following result in the Appendix.
Theorem 4.6  The numeraire portfolio cannot be systematically outperformed by any nonnegative portfolio.

Consequently, no fund manager can systematically outperform the numeraire portfolio. Obviously, if the market portfolio is not the numeraire portfolio and a fund manager approximates the numeraire portfolio, then his or her fund will in the long run outperform the market portfolio.

5 Growth Optimal Portfolio

This section derives the growth optimality of the numeraire portfolio. As in the two asset market, the expected growth $g_{t,h}^δ$ of a strictly positive portfolio $S^δ$ over the time period $(t, t+h]$ is given by the conditional expectation

$$g_{t,h}^δ = E_t \left( \ln \left( \frac{S^δ_{t+h}}{S^δ_t} \right) \right)$$

(5.1)

for all $t, h \geq 0$. To identify the strictly positive portfolio that maximizes the expected growth, one can perturb at time $t \geq 0$, the investment in a given strictly positive portfolio $S^δ \in \mathcal{V}_x^+, x > 0$, by some small fraction $\varepsilon \in (0, \frac{1}{2})$ of some nonnegative portfolio $S^δ$. For analyzing the changes in the expected growth of the perturbed portfolio $S^δ_\varepsilon$, one can define the derivative of expected growth in the direction of $S^δ$ as the right hand limit

$$\left. \frac{\partial g_{t,h}^δ}{\partial \varepsilon} \right|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( g_{t,h}^{\delta_\varepsilon} - g_{t,h}^δ \right)$$

(5.2)

for $t, h \geq 0$. Obviously, if the portfolio that maximizes expected growth coincides in (5.2) with the portfolio $S^δ$, then the resulting derivative of expected growth will always be less than or equal to zero for all nonnegative portfolios $S^δ$. This fact leads to the following alternative definition of growth optimality.

Definition 5.1  A strictly positive portfolio $S^δ$ is called growth optimal if the corresponding derivative of expected growth is always less than or equal to zero for all nonnegative portfolios $S^δ$, that is,

$$\left. \frac{\partial g_{t,h}^δ}{\partial \varepsilon} \right|_{\varepsilon=0} \leq 0$$

(5.3)

for all $t, h \geq 0$.

Note that this definition is different to the classical characterization of a growth optimal portfolio via the maximization of expected logarithmic utility from terminal wealth. This definition via utility was originally used in Kelly (1956) and
later employed in a stream of literature, including Latané (1959), Breiman (1960), Hakansson (1971), Merton (1973a), Roll (1973) and Markowitz (1976), among many others. The following insight is derived in the Appendix by the application of a few basic facts from statistics and probability theory.

**Theorem 5.2**  The numeraire portfolio is growth optimal.

This property of the numeraire portfolio shows again its “best” performance, but from another perspective. It also allows the determination of the numeraire portfolio of a given market model by searching for its growth optimal portfolio, as illustrated for the two asset market.

### 6 Strong Arbitrage

Arbitrage arguments play a central role in the classical *Arbitrage Pricing Theory*, see Ross (1976). However, arbitrage opportunities can only be exploited by market participants. These have to use their portfolios of total tradable wealth. Due to the legal concept of limited liability, negative total wealth of market participants can be neglected. Therefore, a realistic arbitrage concept should focus on nonnegative portfolios. Obviously, a strong form of arbitrage arises when a market participant can generate strictly positive wealth from zero initial capital.

**Definition 6.1**  A nonnegative portfolio $S^δ$ is a strong arbitrage if it starts with zero initial capital, that is $S^δ_0 = 0$, and generates some strictly positive wealth with strictly positive probability at a later time $t \in (0, \infty)$, that is, $P(S^δ_t > 0) > 0$.

The exclusion of the above strong arbitrage in a market has been argued for on purely economic grounds in Loewenstein & Willard (2000). Weaker forms of arbitrage may exist. However, they do not harm the economic viability of the market model. For instance, there may be *free snacks* and *cheap thrills*, in the sense of Loewenstein & Willard (2000). Also *free lunches with vanishing risk* may be present, as discussed in Delbaen & Schachermayer (1998). Such a free lunch can be obtained in the two asset market, when shorting the savings account to fund the fair zero coupon bond. The weaker forms of arbitrage cannot be exploited in practice, since they always require adequate collateral. This refers back to nonnegative portfolios, which are the focus of Definition 6.1. This definition was introduced in Platen (2002), motivated by the supermartingale property. By exploiting this property the following result is derived in the Appendix.

**Theorem 6.2**  There does not exist any nonnegative portfolio that is a strong arbitrage.
This means, strong arbitrage is automatically excluded in the given general setting. Therefore, pricing by excluding strong arbitrage does not make sense. This raises the question: What principle should we instead use for pricing?

7 The Law of the Minimal Price

The supermartingale property (4.4) of a market ensures that the maximum expected return of a nonnegative benchmarked portfolio can at most equal zero. In the case when it equals zero for all time instances, the current benchmarked value of the portfolio is always the best forecast of its future benchmarked values, and one has equality in relation (4.4). Such process is called a martingale, see Shiryaev (1984). As previously mentioned, a price process is called fair if it is a martingale when benchmarked. It must be emphasized that under the benchmark approach not all primary security accounts and portfolios need to be fair, as illustrated in the example. The classical Law of One Price postulates, for a given payoff, that the prices of all replicating portfolios have to be the same. The example indicates that this law is likely to be violated in the US market. This paper proposes a general approach that can handle this case as well.

In a set of nonnegative supermartingales with the same random payoff at a given future time, it is the martingale which always attains the minimal possible value, see Shiryaev (1984). This fundamental fact provides the basis for the following main result of the paper.

Theorem 7.1 (Law of the Minimal Price) If a fair portfolio replicates a given nonnegative payoff at maturity, then this portfolio represents the minimal replicating portfolio among all nonnegative portfolios that replicate this payoff.

The fair replicating portfolio yields the least expensive price process, which is the correct price in a competitive market. The Law of the Minimal Price generates a unique price system in a general setting. Pricing based purely on hedging or no-arbitrage arguments, as proposed under the Arbitrage Pricing Theory, can yield more expensive prices, as illustrated previously.

8 Real World Pricing

Define a contingent claim $H_T$ as a nonnegative payoff, expressed in units of the domestic currency, which is paid at maturity $T \in (0, \infty)$ and has the finite expectation

$$E_0 \left( \frac{H_T}{S_T^\delta} \right) < \infty. \quad (8.1)$$
By the Law of the Minimal Price formulated in Theorem 7.1, one can characterize the least expensive replicating portfolio.

**Corollary 8.1** If for a contingent claim $H_T$, $T \in (0, \infty)$, there exists a fair portfolio $S^\delta_H$ that replicates this claim at maturity $T$ with $H_T = S^\delta_T$, then its minimal price at time $t \in [0, T]$ is given by the real world pricing formula

$$S^\delta_t = S^\delta_t E_t \left( \frac{H_T}{S^\delta_T} \right).$$

(8.2)

The benchmarked fair portfolio value $\hat{S}^\delta_t = \frac{S^\delta_t}{S^\delta_t} = E_t(\frac{H_T}{S^\delta_T})$ forms a martingale, with its current value as the best forecast of its future values. It should be emphasized that the conditional expectation $E_t(\cdot)$ in the real world pricing formula (8.2) is taken under the real world probability measure and the numeraire portfolio $S^\delta_t$ represents the numeraire. No measure transformation is employed. This feature avoids restrictive assumptions that need to be imposed under the classical risk neutral approach. The real world pricing formula (8.2) requires only the existence of the best performing portfolio, the numeraire portfolio, and the finiteness of the expectation in (8.1). Both conditions can hardly be weakened. Obviously, what is driving pricing in (8.2) is the long run performance of the economy.

For establishing the link between real world pricing and standard risk neutral pricing it is helpful to rewrite the real world pricing formula (8.2), for $t = 0$, by employing the benchmarked normalized savings account $\Lambda_T = \hat{S}_T$, obtaining

$$S^\delta_0 = E_0 \left( \Lambda_T \frac{S^0_0}{S^0_T} H_T \right).$$

(8.3)

By the supermartingale property of the normalized benchmark savings account $\Lambda = \{ \Lambda_t = \frac{S^0_t}{S^0_0}, t \geq 0 \}$ with $1 = \Lambda_0 \geq E_0(\Lambda_T)$, equation (8.3) yields the inequality

$$S^\delta_0 \leq \frac{E_0 \left( \Lambda_T \frac{S^0_0}{S^0_T} H_T \right)}{E_0(\Lambda_T)}.$$ 

(8.4)

If the savings account is not fair in a market, then equality cannot be expected in relation (8.4).

However, if one considers the special theoretical case of a market model where the savings account is assumed to be fair, then equality holds in (8.4). In this particular case the expression on the right hand side of (8.4) can be interpreted by Bayes’ formula as the conditional expectation of the discounted contingent claim under some equivalent risk neutral probability measure $Q$ with Radon-Nikodym derivative $\Lambda_T = \frac{dQ}{dP}$. Then relation (8.4) represents the classical risk neutral pricing formula

$$S^\delta_0 = E_0^Q \left( \frac{S^0_0}{S^0_T} H_T \right),$$

(8.5)
see, for instance, Harrison & Kreps (1979), where $E_0^Q$ denotes conditional expectation under $Q$ at time $t = 0$. By inequality (8.4) the fair derivative price is not more expensive than a price obtained under formal application of the standard risk neutral pricing rule. For instance, in the example the fair zero coupon bond is less expensive than the corresponding savings bond. Standard risk neutral pricing requires the existence of an equivalent risk neutral probability measure. The benchmark approach, presented in this paper, removes this restrictive assumption, which can be interpreted more as a mathematical convenience than an economic necessity.

The real world pricing formula (8.2) covers a variety of pricing formulas that appeared in the literature where the Law of one Price is assumed. To mention just a few authors, one may refer to Ingersoll (1987), Constantinides (1992), Duffie (2001) and Cochrane (2001), who use notions such as pricing kernel, state price density, deflator, stochastic discount factor and numeraire portfolio.

Another special case arises when $H_T$ is independent of $S^*_T$. In this case one obtains from the real world pricing formula (8.2) the actuarial pricing formula

$$S_t^{\delta_H} = P(t, T) E_t(H_T)$$

with fair zero coupon bond price $P(t, T)$, as defined in (2.9). The actuarial pricing formula (8.6) exploits the fact that the expectation of a product of independent random variables is the product of their expectations. In the formula (8.6) the fair zero coupon bond $P(t, T)$ provides the discount factor for obtaining the net present value. Thus, the Law of the Minimal Price offers a rigorous and general derivation of the actuarial pricing rule, which has been in use for centuries.

**Conclusion**

The paper derives a general modeling and pricing framework by requiring the existence of a numeraire portfolio, which is in several ways the best performing strictly positive portfolio. In general, the classical Law of One Price no longer holds in this setting. It is replaced by the Law of the Minimal Price, according to which the minimal replicating price process for a given contingent claim is trendless when expressed in units of the numeraire portfolio. It has been demonstrated that, in reality, different self-financing replicating portfolios exist for identical pay-offs. By exploiting the Law of the Minimal Price, extreme maturity derivatives can become significantly less expensive in real markets than currently suggested under classical theories. This finding opens new lines of research and new business perspectives.
9 Appendix

Proof of Theorem 4.4: Consider a strictly positive portfolio $S^δ ∈ V^+_x$, $x > 0$, with the same initial capital as the numeraire portfolio, that is, $S^δ_0 = S^δ_0 = x > 0$. By Corollary 4.3 we can use the following inequality, mentioned in Doob (1953), where for any $k ∈ \{1, 2, \ldots\}$ and $ε ∈ (0, 1)$ one has

$$\exp{\{ε k\} P \left( \sup_{k \leq t < \infty} \hat{S}^δ_t > \exp{\{ε k\}} \right)} \leq E_0 \left( \hat{S}^δ_k \right) \leq \hat{S}^δ_0 = 1.$$  \hspace{1cm} (9.1)

One finds for fixed $ε ∈ (0, 1)$ that

$$\sum_{k=1}^{∞} P \left( \sup_{k \leq t < \infty} \ln \left( \hat{S}^δ_t \right) > ε k \right) \leq \sum_{k=1}^{∞} \exp{-ε k} < ∞.$$

By the Lemma of Borel and Cantelli, see Shiryaev (1984), there exists a random variable $k_ε$ such that for all $k ≥ k_ε$ and $t ≥ k$ it holds that

$$\ln \left( \hat{S}^δ_t \right) ≤ ε k ≤ ε t.$$

Therefore, it follows for all $k > k_ε$ the estimate

$$\sup_{t≥k} \frac{1}{t} \ln \left( \hat{S}^δ_t \right) ≤ ε,$$

which implies that

$$\limsup_{t→∞} \frac{1}{t} \ln \left( \frac{\hat{S}^δ_t}{\hat{S}^δ_0} \right) ≤ \limsup_{t→∞} \frac{1}{t} \ln \left( \frac{\hat{S}^δ_t}{\hat{S}^δ_0} \right) + ε.$$  \hspace{1cm} (9.2)

Since the inequality (9.2) holds for all $ε ∈ (0, 1)$ one obtains with (4.5) the relation (4.6) \hspace{1cm} □

Proof of Theorem 4.6: Consider a nonnegative portfolio $S^δ$ with benchmarked value $\hat{S}^δ_t = 1$ at a given time $t ≥ 0$, where $\hat{S}^δ_t ≥ 1$ at some time $s ∈ [t, ∞)$. Then it follows by Corollary 4.3, that

$$0 ≥ E_t \left( \hat{S}^δ_s - \hat{S}^δ_t \right) = E_t \left( \hat{S}^δ_t - 1 \right) ≥ 0.$$  \hspace{1cm} (9.3)

Since one has $\hat{S}^δ_s ≥ 1$ and $E_t(\hat{S}^δ_s) ≤ 1$, it can only follow that $\hat{S}^δ_s = 1$. This means that one has at time $s$ the equality $S^δ_s = S^δ_s$. Therefore, according to Definition 4.5, the portfolio $S^δ$ does not systematically outperform the numeraire portfolio. \hspace{1cm} □
Proof of Theorem 5.2: For two consecutive times \( t \) and \( t + h, \ h > 0; \ \varepsilon \in (0, \frac{1}{2}) \); and a nonnegative portfolio \( S^\delta_t \) with \( S^\delta_t > 0 \), one considers the perturbed portfolio \( S^\delta \) with the choice \( S^\delta_t = S^\delta_t \) in (5.2), yielding a portfolio ratio \( A^\delta_{t,h} = \varepsilon A^\delta_{t,h} + (1 - \varepsilon) A^\delta_{t,h} > 0 \). One then obtains by the well-known inequality \( \ln(x) \leq x - 1 \) for \( x \geq 0 \), the relations

\[
G^\delta_{t,h} = \frac{1}{\varepsilon} \ln \left( \frac{A^\delta_{t,h}}{A^\delta_{t,h}} \right) \leq \frac{1}{\varepsilon} \left( \frac{A^\delta_{t,h}}{A^\delta_{t,h}} - 1 \right) = \frac{A^\delta_{t,h} - 1}{A^\delta_{t,h}}
\]

and

\[
G^\delta_{t,h} = -\frac{1}{\varepsilon} \ln \left( \frac{A^\delta_{t,h}}{A^\delta_{t,h}} \right) \geq -\frac{1}{\varepsilon} \left( \frac{A^\delta_{t,h}}{A^\delta_{t,h}} - 1 \right) = \frac{A^\delta_{t,h} - A^\delta_{t,h}}{A^\delta_{t,h}}.
\]

Because of \( A^\delta_{t,h} > 0 \) one obtains from (9.5) for \( A^\delta_{t,h} - A^\delta_{t,h} \geq 0 \) the inequality

\[
G^\delta_{t,h} \geq 0,
\]

and for \( A^\delta_{t,h} - A^\delta_{t,h} < 0 \) because of \( \varepsilon \in (0, \frac{1}{2}) \) and \( A^\delta_{t,h} \geq 0 \) the relation

\[
G^\delta_{t,h} \geq -\frac{1}{1 - \varepsilon} \geq -\frac{1}{1 - \varepsilon} \geq -2.
\]

Summarizing (9.4)–(9.7) yields the upper and lower bounds

\[
-2 \leq G^\delta_{t,h} \leq \frac{A^\delta_{t,h}}{A^\delta_{t,h}} - 1,
\]

where by Definition 4.1

\[
E_t \left( \frac{A^\delta_{t,h}}{A^\delta_{t,h}} \right) \leq 1.
\]

By using (9.8) and (9.9) it follows by the Dominated Convergence Theorem, see Shiryaev (1984), that

\[
\frac{\partial g^\delta_{t,h}}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \lim_{\varepsilon \to 0^+} E_t \left( G^\delta_{t,h} \right) = E_t \left( \lim_{\varepsilon \to 0^+} G^\delta_{t,h} \right) = E_t \left( \frac{A^\delta_{t,h}}{A^\delta_{t,h}} \right) - 1.
\]

This proves by (4.1) and Definition 5.1 that the numeraire portfolio \( S^\delta \) is growth optimal. \( \square \)
Proof of Theorem 6.2: Assume, without loss of generality, that $S^\delta \in \mathcal{V}_x^+$, $x > 0$. For a nonnegative portfolio $S^\delta$, which starts with zero initial capital, it follows by Corollary 4.3 that

$$0 = S^\delta_0 = x \hat{S}^\delta_0 \geq x E_0 \left( \hat{S}^\delta_t \right) = x E(\hat{S}^\delta_t) \geq 0,$$

for $t \geq 0$, where $E(\cdot)$ denotes expectation. By the nonnegativity of $S^\delta_t$ and the strict positivity of $S^\delta_0$, the event $S^\delta_t > 0$ can only have zero probability, that is

$$P \left( S^\delta_t > 0 \right) = 0. \quad (9.11)$$

This leads to the conclusion that $S^\delta_t$ equals zero for all $t \geq 0$, which proves by Definition 6.1 the Theorem 6.2.

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