Liability Driven Investments under a Benchmark Based Approach

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Abstract. In this paper, we present an alternative approach as a suitable framework under which liability driven investments can be valued and hedged. This benchmark approach values both assets and liabilities consistently under the real world probability measure using the best performing portfolio, the growth optimal portfolio, as benchmark and numéraire. The benchmark approach identifies the investment strategy which is replicating a given claim at minimal cost. Should the liability under consideration be subject to nonhedgeable risk, e.g. mortality risk, benchmarked risk minimization identifies with its real world pricing formula the investment strategy which minimizes in a practical sense the price of a given claim and minimizes the benchmarked profit and loss from hedging. The application of the approach will be demonstrated for pensions. A least expensive pension scheme will be described that allows one in a fair and transparent manner to hedge in the least expensive way with minimal risk the post retirement payments for its members.

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1 Introduction

Liability Driven Investment (LDI) can be understood as the design and implementation of investment strategies whose objective is to ensure that assets meet liabilities, we refer to Working Party (2007) for a typical discussion on this topic and to Blackrock (2012) for a discussion on the recent performance of LDI. LDI is highly important for pension funds and insurance. The topic has enjoyed considerable attention in recent years, especially after the Pensions Act 1995, and even more so the report by Myners (2001), which encouraged trustees to take carefully into account their liabilities when deciding on asset allocation. Prior to this, trustees implemented primarily balanced investment strategies subject to a reference index, which was usually designed to meet the median or mean return achieved by a peer group. This competitive pressure resulted over time in an increase in the proportion of assets invested in equities until this development was curbed by the Pensions Act 1995.

In Working Party (2007), the authors focus on the implementation of LDI strategies from a practical point of view. In particular, they draw the trustees’ attention to the fact that the problem of choosing an appropriate discount factor for the valuation of liabilities is not yet completely solved, albeit their view is presented and supported by a discussion. Secondly, the Working Party (2007) discusses in great detail a split between matching and return seeking assets. Although the appropriateness of different strategies and asset classes is discussed, taking financial market considerations into account, the responsibility of choosing a particular strategy remains vested with the trustees. What is lacking is a solid theoretical foundation, which guarantees that the investment strategies meet the liabilities in the least expensive way and where the exposure to nonhedgeable risk is minimized in an appropriate manner.

This paper offers a systematic alternative solution to the above problem. It employs a benchmark based approach, the benchmark approach developed in Platen (2002) and Platen & Heath (2010). The benchmark is the best performing portfolio, which is the growth optimal portfolio (GOP) or numéraire portfolio of the given investment universe. The latter has received for a long time attention in the finance literature, see e.g. Kelly (1956), Latané (1959), Breiman (1960), Thorp (1961), Markowitz (1976), Long (1990), and MacLean, Thorp & Ziemba (2011). In particular, the GOP is the portfolio that maximizes logarithmic utility from terminal wealth. Due to the optimal growth of the portfolio, any other nonnegative portfolio when denominated in units of the GOP, turns out to have in the long-run zero or negative drift.

Regarding the first issue raised in Working Party (2007), that is the valuation of liabilities, the benchmark approach values assets and liabilities consistently in terms of this optimally growing portfolio using real world expectations. They turn out to be less expensive than when valued under the classical risk neutral approach, see Platen (2009). This for many surprising fact is illustrated in Figure
1.1, which will be explained later in detail. It compares the evolution of the logarithm of a zero coupon bond, with maturity at the end of the time period priced under the classical risk neutral approach with the logarithm of a zero coupon bond calculated via the real world pricing formula of the benchmark approach.

We point out that nonnegative assets, when denominated in terms of the benchmark are supermartingales. This means, their current benchmarked value is equal or greater than their future expected benchmarked values. Furthermore, the benchmark approach allows long-term investments to exploit systematically the observed magnitude of the equity premium. As fixed-income investments are outperformed by equity investments in the long-run, liabilities are least expensively valued in terms of the benchmark. They cannot be hedged with minimal expenses by solely relying on fixed-income products. Benchmarking risk minimization, see Du & Platen (2012), systematically provides the least expensive hedge with, in a practical sense, minimal hedge error, as we will demonstrate in this paper. This addresses the second issue raised in Working Party (2007).

We will see that under the benchmark approach, liabilities that pay at their maturity dates units of the savings account are hedged by investing funds in both, the benchmark and the savings account. The hedge is performed in a manner which ensures that the liability is met. This type of well functioning new hedge is illustrated in Figure 1.1, for the model that we calibrate later in Section 5. It shows in its lower graph the evolution of the logarithm of the value of the hedge portfolio and price. As the trajectories of the logarithms of price and hedge portfolio are visually indistinguishable, we only plot the former. The logarithm of the value implied by risk neutral valuation is shown in Figure 1.1 in the upper graph. The hedge portfolio we obtain under the benchmark approach matches the liability, it does so at a substantially lower cost than the risk neutral counterpart. We will find for the choice of the benchmark in this paper that the benchmark approach reduces, for instance, the cost of a 35 years to maturity zero coupon bond by about 65%, which is a major difference that is illustrated in Figure 1.1. As the time to maturity approaches the difference vanishes. The benchmark approach achieves this reduction in cost by employing the benchmark as a ”matching asset”, using the terminology from Working Party (2007). The benchmark approach does not distinguish between matching and return seeking assets. Instead, the savings account and benchmark are employed in hedging to ensure that the hedge meets the claim.

For the hedge of the zero coupon bond the benchmark is employed similarly as in common financial planning, however, in a rigorous quantitative manner yielding the highest possible payouts. This is illustrated in Figure 1.2, which shows the fraction of wealth invested in the benchmark: Initially, a large proportion of the value of the hedge portfolio is invested in the benchmark, so that it can exploit the equity premium over some time. Additionally, we will see that the hedge buys equity when its value is low and sells it when it is high. As the
Figure 1.1: Comparison of the logarithms of the prices of a zero-coupon under risk-neutral pricing and real world pricing.

As the maturity of the zero coupon bond draws closer, the entire hedge portfolio for the zero coupon bond is finally invested in the savings account. In this way, the liability is met irrespective of equity market fluctuations. In summary, in a mathematically rigorous manner the higher long-term growth of the benchmark over the savings account is systematically exploited to achieve a significantly less expensive replication of the liability than feasible under risk neutral pricing.

Not fully hedgeable liabilities, such as mortality-linked liabilities, which occur in life insurance and pension schemes, are more difficult to manage, since part of the risk is not traded. Consequently, a mortality-linked zero coupon bond, which pays one dollar in the event that the owner survives until maturity, cannot be perfectly hedged. However, mortality risk can be pooled in an insurance company or in a pension scheme and, thus, mostly diversified. To implement this pooling systematically, one can employ the already mentioned benchmarked risk minimization, which provides us with three outcomes: First, it identifies the minimal possible amount that has to be invested for hedging the claim when the nonhedgeable risk is diversified. Second, it provides a self-financing investment strategy allowing us to hedge the hedgeable part of the claim. Finally, fluctuations of the benchmarked profit and loss (P&L) are minimized in a practical sense made precise in Section 3.

The above indicated minimizations of costs and risks are highly desirable from the perspective of the members and trustees of a pension scheme. We point out that by pooling a sufficiently large number of independent benchmarked P&L processes, the resulting total benchmarked P&L process can become practically negligible. Unfortunately, there are uncertainties faced by the pension schemes that cannot be hedged or diversified. These concern, for instance, model errors,
errors from discrete hedging, changes in regulation, increases in longevity beyond current estimates, etc. To cover these uncertainties the paper suggests to design least expensive pensions. On one hand they shall exploit the above discussed minimal possible prices. On the other hand, they shall share all nonhedgeable uncertainties in a fair and transparent manner. Using real world pricing of the benchmark approach, under the available information at a given time, will achieve this. When sharing the nonhedgeable risks and potential gains in a pension scheme, such a mutual scheme is most likely in the long run less expensive than schemes that use an external provider like an insurance company or a provider of life annuities. We will provide an example which indicates how least expensive pension schemes can be designed.

The structure of the paper is as follows: In Section 2, we explain the benchmark approach to contingent claim pricing. Section 3 considers the pricing and hedging of liabilities. Least expensive pensions are introduced in Section 4. Section 5 presents a case study, demonstrating how the methodology, proposed in this paper, can be employed by a least expensive pension scheme.

2 A Benchmark Framework

The aim of this section is to show that the benchmark approach can identify the investment strategy that hedges at the minimal possible cost a given financial liability.

To model uncertainty we use a filtered probability space \((\Omega, \mathcal{A}, \mathcal{F}, P)\), where the filtration \(\mathcal{F} = (\mathcal{A}_t)_{t \in [0, \infty)}\) is assumed to satisfy the usual conditions, see Karatzas & Shreve (1998). We introduce the savings account, \(B_t\), satisfying the differential
equation

\[ dB_t = r_t B_t dt, \]

for \( t \geq 0 \), with \( B_0 = 1 \), where \( r = \{ r_t, t \geq 0 \} \) is the \( \mathcal{F} \)-adapted short rate process. We now introduce the benchmark \( S^* = \{ S^*_t, t \geq 0 \} \), which is the numéraire portfolio and also the growth optimal portfolio (GOP). It has been shown, see e.g. Platen & Heath (2010), that in a continuous financial market the dynamics of the GOP, that is the benchmark, are given by the stochastic differential equation (SDE)

\[ dS^*_t = S^*_t (r_t dt + \theta_t (\theta_t dt + dW_t)), \]

where the volatility \( \theta_t \) denotes the market price of risk. The source of traded uncertainty, which drives the benchmark is the standard Brownian motion \( W = \{ W_t, t \geq 0 \} \). Equivalently, the dynamics of the discounted GOP \( \bar{S}_t = \frac{S^*_t}{B_t} \) are given by the SDE

\[ d\bar{S}_t = \bar{S}_t \theta_t (\theta_t dt + dW_t). \]

To simplify our presentation we will focus on investment strategies

\[ \delta = \{ \delta_t = (\delta^0_t, \delta^1_t), t \in [0, \infty) \}, \]

which at time \( t \) invest \( \delta^0_t \) units in the savings account \( B_t \) and \( \delta^1_t \) units in the benchmark \( S^*_t \). Note that more general strategies can be handled in a similar manner, see Platen & Heath (2010). We make the technical assumptions that \( \delta \) is a predictable stochastic process and the Itô integral

\[ \int_0^t \delta^1_udS^*_u \]

exists expressing the gains from trading in the benchmark. Given a strategy \( \delta \), the value of the corresponding portfolio \( S^\delta = \{ S^\delta_t, t \geq 0 \} \) is given at time \( t \) by

\[ S^\delta_t = \delta^0_t B_t + \delta^1_t S^*_t. \]

We call a strategy \( \delta \) and the corresponding portfolio \( S^\delta \) self-financing if

\[ dS^\delta_t = \delta^0_t dB_t + \delta^1_t dS^*_t, \quad (2.1) \]

for \( t \geq 0 \). Intuitively, the SDE (2.1) means that the value of the self-financing portfolio only changes due to gains or losses from trading in the savings account and the benchmark. Funds are neither injected into nor withdrawn from a self-financing portfolio. We have the following important proposition, which highlights the central role of the numéraire portfolio as benchmark:

**Proposition 2.1** Let \( S^\delta \) be a nonnegative self-financing portfolio. Then the corresponding benchmarked portfolio \( \hat{S}^\delta = \{ \hat{S}^\delta_t = \frac{S^\delta_t}{S^*_t}, t \geq 0 \} \) is a supermartingale, that is,

\[ \hat{S}^\delta_t \geq E \left( \hat{S}^\delta_s | \mathcal{A}_t \right) \quad (2.2) \]

for all \( 0 \leq t \leq s < \infty \).
Proof: Recall our standing assumption that $\delta$ and $S^\delta$ are assumed to be self-financing. Now, by the Itô-formula it follows

$$d \left( \hat{S}^\delta_t \right) = \frac{d \left( S^\delta_t \right)}{S^\delta_t} + S^\delta_t \left( \frac{1}{S^\delta_t} \right) d \left( S^\delta_t \right) + d\langle S^\delta, \frac{1}{S^\delta} \rangle_t$$

$$= -\delta^0_t \hat{B}_t \theta_t dW_t,$$

where $\hat{B}_t = \frac{B_t}{S^\delta_t}$ is the benchmarked savings account. Hence $\hat{S}^\delta$ is driftless and, thus, a local martingale. Since by assumption $S^\delta$ is nonnegative, so is $\hat{S}^\delta$. It is well-known that a nonnegative local martingale is a supermartingale. Hence, $\hat{S}^\delta$ is a supermartingale, which completes the proof. \(\square\)

The conclusion of Proposition 2.1 is crucial: It means that any nonnegative self-financing portfolio, when expressed in units of the benchmark, is a supermartingale. Intuitively, in the long-run the benchmarked portfolio value is either in the mean downward trending or has zero trend. Roughly speaking the portfolio $S^\delta$ does not grow in the long-run faster than the GOP, which is intuitive, as the latter is in many ways the best performing portfolio. This fact has the following important implication for liability driven investment: Consider a liability that falls due at time $T$. We denote its value by $L_T$, and its benchmarked value by $\hat{L}_T = \frac{L_T}{S_T^\delta}$. Assume now that we find an investment strategy $\delta_{L_T}$, which replicates perfectly the claim of the liability at maturity, that is, we have

$$\hat{L}_T = \hat{S}^\delta_{L_T} T.$$

From Proposition 2.1, it follows immediately that the value of the portfolio at time $t \leq T$ satisfies the relation

$$\hat{S}^\delta_{L_T} t \geq E \left( \hat{S}^\delta_{L_T} T | A_t \right) = E \left( \frac{L_T}{S_T^\delta} | A_t \right). \quad (2.3)$$

This inequality leads to the important conclusion that if we have equality in relation (2.3), i.e.

$$\hat{S}^\delta_{L_T} t = E \left( \hat{S}^\delta_{L_T} T | A_t \right), \quad (2.4)$$

then the strategy $\delta_{L_T}$ is the least expensive investment strategy, that can replicate the liability $L_T$. This statement is also referred to as the Law of the Minimal Price, see e.g. Platen (2009) and Platen & Bruti-Liberati (2010). Investment strategies resulting in minimal prices are the investment strategies we are interested in, as they allow us to replicate a given liability at the lowest possible cost. The associated price processes are referred to as fair price processes. One can argue that in a competitive market, the least expensive price, which is the fair price, is also the economically correct price.

For a benchmarked fair price process its current benchmarked value is the best forecast of its future benchmarked values, and thus forms an $(\mathcal{A}, P)$-martingale;
see Karatzas & Shreve (1998). Equality (2.4) is equivalent to the real world pricing formula
\[ S_t^{\delta_{LT}} = S_t^\ast E \left( \frac{L_T}{S_t^T} | A_t \right), \]
for \( 0 \leq t \leq T < \infty \), see Platen & Heath (2010) and Platen & Bruti-Liberati (2010). We alert the reader to the fact that different to risk neutral pricing, under real world pricing we compute in (2.5) the expectation under the real world probability measure using the benchmark as numéraire.

2.1 Strategies for Savings Account Liabilities

In the following, we will illustrate how, in practice, we can find investment strategies which replicate a given contingent claim in the least expensive way. To illustrate real world pricing of fixed-income liabilities we introduce a reasonably realistic and highly tractable model for the GOP as benchmark, namely the minimal market model (MMM), see e.g. Platen (2001) and Platen & Heath (2010). We remark that empirically the MMM has been successfully applied to the pricing and hedging of long-dated contingent claims; see Hulley & Platen (2012) and Platen & Bruti-Liberati (2010). The discounted GOP is assumed to satisfy the following equation
\[ \tilde{S}_t = \alpha_t Y_t. \]
Here \( Y = \{ Y_t, t \geq 0 \} \) is a squared Bessel process of dimension four, given by the SDE
\[ dY_t = (1 - \eta Y_t) dt + \sqrt{Y_t} dW_t, \]
where \( Y_0 > 0 \). Furthermore, \( \alpha_t \) is a deterministic function of time,
\[ \alpha_t = \alpha_0 \exp \{ \eta t \}. \]
We point out that \( \alpha_t \) equals the drift of the discounted GOP and \( \theta_t = \frac{1}{\sqrt{Y_t}} \) its volatility. Furthermore, the exponential function \( \alpha_t \) models the long-term average growth of the economy, with \( \eta \) representing its long-term net growth rate.

In the current paper, we will focus, for simplicity, on the pricing and hedging of contingent claims paying units of the savings account at some fixed dates. More precisely, we are interested in liabilities with streams of payoffs, where the single payoffs at given maturity dates are multiples of the value of the savings account \( B_T \) contingent on certain events. For instance, in the context of post-retirement pension payments, such type of claims are easily seen to be of importance: They can be interpreted as interest-indexed payments. As the short rate can be expected to exceed on average the inflation rate, see Dimson, Marsh & Staunton (2002), one can expect payoffs that are, on average, slightly in excess of the inflation rate. We remark that interest-indexed annuities pay a stream of savings account payoffs at respective maturities. Hence, in the following we focus on
savings account payoffs as building blocks for annuities, as well as, life annuities. We will see that we do not need in this case to model the interest rate.

The next proposition provides the minimal price and the corresponding fair hedge portfolio process for a savings account payoff $B_T$. Under the given model, by using the real world pricing formula, we are assured that we identify the least expensive way of replicating the savings account liability $B_T$.

**Proposition 2.2** Let $\hat{B}_t = \frac{B_t}{S_t} = \frac{1}{S_t}$ denote the value of the benchmarked savings account at time $t \geq 0$. Under the MMM, the discounted real world price at time $t \in [0, T]$ for the savings account payoff $B_T$, $0 < T < \infty$, is given by the formula

$$Q(t, T) := \frac{1}{\hat{B}_t} E\left( \hat{B}_T | \mathcal{A}_t \right)$$

$$= \left( 1 - \exp \left( -\frac{1}{2} \left( \hat{B}_t \right)^{-1} \left( \varphi(T) - \varphi(t) \right) \right) \right)$$

(2.6)

with

$$\varphi(t) = \varphi(0) + \frac{1}{4} \int_0^t \alpha_s ds,$$  

(2.7)

$$= \varphi(0) + \frac{\alpha_0}{4\eta} (\exp \{ \eta t \} - 1)$$

(2.8)

and the strategy for the fair replicating portfolio is given by

$$\delta^0_t = \frac{\partial \hat{Q}(t, T, \hat{B}_t)}{\partial \hat{B}_t} , \delta^1_t = \hat{Q}(t, T, \hat{B}_t) - \delta^0_t \hat{B}_t.$$

Here $\hat{Q}(t, T, \hat{B}_t) = \frac{Q(t, T)}{S_t}$ denotes the benchmarked price at time $t$ for one unit of the savings account at maturity $T$.

**Proof:** Equality (2.6) follows by application of the transition density of the square-root process $Y$, see e.g. Platen & Heath (2010). It is clear that $\hat{Q}(t, T, \hat{B}_t) = E \left( \hat{B}_T | \mathcal{A}_t \right)$ is a martingale, hence

$$\hat{Q}(t, T, \hat{B}_t) = \hat{Q}(0, \hat{B}_0) + \int_0^t \frac{\partial \hat{Q}(u, T, \hat{B}_u)}{\partial \hat{B}_u} d\hat{B}_u,$$

(2.9)

which shows the hedge ratio

$$\delta^0_t = \frac{\partial \hat{Q}(t, T, \hat{B}_t)}{\partial \hat{B}_t}.$$
Finally, we use the relation
\[ \hat{Q}(t, T, \hat{B}_t) = \delta_0^t \hat{B}_t + \delta_1^t, \]
to complete the proof. □

Proposition 2.2 gives us the minimal price and the corresponding investment strategy by which the liability can be hedged. From (2.6), it is clear that the investment strategy is fully determined by the payoff of the liability. It uses both, the savings account and the benchmark to replicate the claim \( B_T \). Note that in formula (2.6) for the price of the savings account liability, the particular dynamics of the short rate do not matter. This simplifies our analysis and makes pricing and hedging robust towards the modeling of the short rate dynamics.

A reader acquainted with risk neutral pricing would have expected the price of the claim at time \( t \) to be \( \hat{B}_t \), and the hedging strategy to purchase one unit of the savings account and hold it until time \( T \). Obviously, this strategy replicates the claim. However, it does so at a cost that can be substantially higher than the one presented in Proposition 2.2. To emphasize the ideas introduced in this section, we formalize this discussion by considering the investment strategy \( \tilde{\delta}_0^t = 1, \tilde{\delta}_1^t = 0 \), which the risk neutral approach would have proposed. Of course, this portfolio is self-financing and its value at time \( t \) is that of the savings account
\[ S_t^\delta = \tilde{\delta}_0^t B_t + \tilde{\delta}_1^t S_t^* = B_t. \]

Furthermore, under the MMM the dynamics of the benchmarked savings account value \( \hat{B}_t \) are given by the SDE
\[ d\hat{B}_t = -\alpha_t \left( \hat{B}_t \right)^{\frac{3}{2}} dW_t, \]
so that \( \hat{B} = \{ \hat{B}_t, t \geq 0 \} \) is a nonnegative local martingale, and thus a supermartingale. Important to note is that \( \hat{S}_t^\delta = \hat{B}_t \) forms a strict supermartingale, where by (2.6) we have
\[ \hat{S}_t^\delta = \hat{B}_t > \hat{B}_t \left( 1 - \exp \left\{ -\frac{\hat{B}_t^{-1}}{2(\varphi(T) - \varphi(t))} \right\} \right) = \hat{Q}(t, T, \hat{B}_t) \hat{B}_t = E \left( \hat{S}_T^\delta | \mathcal{A}_t \right), \]
for \( 0 \leq t \leq T \). Consequently, the portfolio \( S_t^\delta \) does not offer the least expensive way of replicating the savings account liability \( B_T \).

We remark that by the same argument that the benchmarked savings account is not a martingale, it can be shown that under the MMM the Radon-Nikodym derivative of the putative risk neutral measure is not a martingale. Therefore, the MMM does not admit an equivalent risk neutral probability measure, and risk neutral pricing does not apply to this model. Still, the real world pricing formula identifies the fair portfolio with the investment strategy that replicates \( B_T \) at the minimal possible price.
Let us briefly summarize the main ideas introduced so far: Under the real world probability measure nonnegative benchmarked portfolios are supermartingales. Consequently, if we can find a fair portfolio, which replicates the liability under consideration, we are assured that this is the investment strategy that replicates the claim at the minimal possible cost. The minimal price is uniquely identified, using the real world pricing formula. Other pricing rules can result in different prices and investment strategies, but all these will be more expensive due to the supermartingale property of benchmarked securities.

3 Investment Strategies for Mortality-Linked Liabilities

In this section, we show how to extend the approach, presented in the previous section, to liabilities that are subject to nonhedgeable risk. In particular, we consider mortality risk under the assumption that this risk can be modeled and is not traded.

On the filtered probability space \((\Omega, \mathcal{A}, \mathcal{F}, P)\) we introduce the random variable \(\xi_x\), which denotes the age at death of an individual aged \(x\) at time 0. We assume that \(\xi_x\) is independent of \(\bar{S}\), the discounted GOP process. Let \(1_A\) denote the indicator function, which takes the value one if \(A\) is true and zero otherwise. Regarding \(\xi_x\), we make below similar distributional assumptions as in Li, Hardy & Tan (2008), who employ threshold lifetables. Consider an individual aged 0 at time 0, and fix a threshold age \(N\). On the interval \([0, N]\), \(\xi_0\) is assumed to follow a Gompertz distribution, i.e.

\[
P(\xi_0 > t) = G^0(t) = \exp \left\{ - \frac{b}{\ln(c)} \left( c^t - 1 \right) \right\},
\]

for \(t \in [0, N]\) and \(b > 0, c > 1\). For \(t > N\), Li, Hardy & Tan (2008) invoke the Balkema-de Haan-Pickands Extreme Value Theorem, see McNeil, Frey & Embrechts (2005), which states that under some regularity conditions, the limiting extreme value distribution of

\[
Z = \xi_0 - N,
\]

for \(\xi_0 > N\) follows the generalized Pareto distribution, as \(N\) tends to the right-hand side of the support of \(\xi_0\). Consequently, we set

\[
P(\xi_0 > t) = G^0(t) = p \left( 1 + \gamma \left( \frac{t - N}{\vartheta} \right) \right)^{-\frac{1}{\gamma}},
\]

where \(p = G^0(N)\) and \(\vartheta > 0\). Note that besides a theoretical justification stemming from extreme value theory, an additional advantage of the use of threshold lifetables is their computational tractability. Finally, we set

\[
G^x(t) = P(\xi_x > t) = \frac{G^0(t + x)}{G^0(x)}.
\]
For simplicity, we focus on the payoff $1_{\xi_x > T}B_T$, a mortality-linked savings account payoff at time $T$. This type of liability is relevant to pension schemes and insurance contracts. Under the above assumption, LDI becomes a more difficult task: One can think of a claim as consisting of a part that can be hedged in the financial market via a self-financing trading strategy and a part that is non-hedgeable. The presence of the nonhedgeable component, which is here due to mortality risk, makes the pricing problem more difficult in the following sense: In Section 2, our objective was to find the minimal cost at which a given claim can be replicated. However, the presence of the nonhedgeable part means that this criterion no longer applies, as the claim cannot be perfectly hedged. The above problem can be systematically solved using the concept of benchmarked risk minimization (BRM), which was introduced in Du & Platen (2012). BRM identifies the investment strategy which has the lowest set-up cost and minimizes the fluctuations of the benchmarked non-hedgeable part, in a sense we make precise below. From the point of view of a large pension scheme, interested in finding an investment strategy that can hedge a particular liability, it is important that it can do so at a competitive initial price, the first property of the strategy that is identified by BRM. Additionally, it is desirable that fluctuations in the benchmarked nonhedgeable part of the claim are minimal, to avoid financial distress of the scheme due to short-term fluctuations of the benchmarked nonhedgeable part. This will be achieved by the second property of the hedging strategy identified by BRM, see Du & Platen (2012).

We now formalize BRM and apply it to the problem of hedging a mortality-linked savings account payoff. Assuming all benchmarked price processes to be supermartingales it follows under BRM that the minimal benchmarked price process forms a martingale. This yields the real world pricing formula (2.5) for nonhedgeable contingent claims. This allows us to obtain prices for payoffs subject to nonhedgeable mortality risk for our example. It provides us with the benchmarked price process $\hat{V} = \{V_t = 1_{\xi_x > t}B_t, t \geq 0\}$ given by the real world conditional expectation,

$$\hat{V}_t := E \left( 1_{\xi_x > T}B_T | \mathcal{A}_t \right).$$

(3.11)

It follows from the assumed independence of $\xi_x$ and $B_T$ an explicit formula for the minimal possible benchmarked price of the above payoff in the form

$$\hat{V}_t = \frac{G^x(T)}{G^x(t)} \hat{B}_t | \mathcal{A}_t.$$

(3.12)

As discussed above, we cannot expect to find a self-financing trading strategy $\delta = \{\delta_t, t \geq 0\}$ so that its associated portfolio $S^\delta$ replicates perfectly the payoff $1_{\xi_x > T}B_T$. Some "hedge error" will arise. Following BRM, we introduce the benchmarked P&L process $\hat{C} = \{\hat{C}_t, t \in [0, T]\}$, defined by

$$\hat{C}_t = \hat{V}_t - \int_0^t \delta_u d\hat{B}_u - \hat{V}_0.$$

(3.13)
Since we require that $\delta = \{\delta_t, t \geq 0\}$ is a self-financing strategy, we have
\[
\delta^1_t = \hat{V}_t - \hat{C}_t - \delta^0_t \hat{B}_t.
\]
We also remark that when setting up the hedge at the initial time $t = 0$ we assume zero benchmarked P&L, that is $\hat{C}_0 = 0$, an important property as we will discuss below. Regarding equation (3.13), we have
\[
\hat{V}_t = \hat{V}_0 + \int_0^t \delta^0_u d\hat{B}_u + \hat{C}_t,
\]
i.e. we decompose the benchmarked price $\hat{V}_t$ into an initial premium $\hat{V}_0$, a self-financing part $\int_0^t \delta^0_u d\hat{B}_u$ and the remaining benchmarked P&L term $\hat{C}_t$. BRM now allows us to find the self-financing trading strategy $\delta = \{\delta_t, t \geq 0\}$ for which, firstly the initial price $V_0 = \hat{V}_0 S^*_0$ is minimal, and secondly the benchmarked P&L process $\hat{C}$ and any benchmarked self-financing portfolio are orthogonal, that is, the respective product forms a local martingale. This means in our setting that this product is driftless. Intuitively, this means that the benchmarked P&L, interpreted as "distance" between the benchmarked contingent claim and benchmarked traded uncertainty is minimized. In Du & Platen (2012), it is illustrated that under sufficiently strong integrability and independence assumptions BRM minimizes the second moment of the benchmarked P&L $\hat{C}_T$.

It is convenient to identify a corresponding martingale representation of $\hat{V}_T$. Below, we will decompose $\hat{V}_T$ as follows:
\[
\hat{V}_T = \hat{V}_0 + M^1_T + M^2_T,
\]
where $M^1 = \{M^1_t, t \in [0, T]\}$ and $M^2 = \{M^2_t, t \in [0, T]\}$ are local martingales, which correspond to the benchmarked self-financing part and the benchmarked nonhedgeable part of the claim, respectively. We associate with $\xi_x$ the process
\[
H^x_t = 1_{\xi_x > t},
\]
which indicates whether the individual is alive at time $t$. This leads to the following result.

**Proposition 3.1** The martingale representation for
\[
\hat{V} = \{\hat{V}_t = \hat{V}(t, T, \hat{B}_t), t \geq 0\}
\]
has under BRM the form
\[
\hat{V}_t = E\left( H^x_T \hat{B}_T \right) + \int_0^t \frac{\partial \hat{V}(u, T, \hat{B}_u)}{\partial B_u} d\hat{B}_u + \int_0^t \hat{V}_u d\left( \frac{H^x_u}{G^x(u)} \right),
\]
where
\[
\hat{V}_t = \hat{V}(t, T, \hat{B}_t) = \frac{G^x(T)}{G^x(t)} H^x_t Q(t, T, \hat{B}_t),
\]
13
and \( \dot{Q}(t, T, \dot{B}_t) = \frac{Q(t, T)}{S_t} \) follows from (2.6). Furthermore, \( M^1 = \{ M^1_t, t \in [0, T] \} \) and \( M^2 = \{ M^2_t, t \in [0, T] \} \), are given by the local martingales

\[
M^1_t := \int_0^t \frac{\partial \dot{V}(u, T, \dot{B}_u)}{\partial B_u} d\dot{B}_u,
\]

and

\[
M^2_t := \int_0^t \dot{V}_u d \left( \frac{H^x_u}{G^x(u)} \right),
\]

respectively, which are orthogonal in the sense that their product is driftless.

**Proof:** Recall from (3.11) and the assumed independence of \( H^x_T \) and \( S_T \), it follows (3.12) such that

\[
\dot{V}_t = \frac{G^x(T)}{G^x(t)} H^x_t \dot{Q}(t, T, \dot{B}_t).
\]

We note from (2.9) that the covariation of the processes

\[
\left\{ \frac{H^x_t}{G^x(t)}, t \geq 0 \right\} \text{ and } \left\{ \dot{Q}(t, T, \dot{B}_t), t \geq 0 \right\}
\]

vanishes. Hence from the Itô-formula, (2.9), and Proposition 2.2, we have

\[
d\dot{V}_t = \frac{G^x(T)}{G^x(t)} H^x_t d\dot{Q}(t, T, \dot{B}_t) + G^x(T) \dot{Q}(t, T, \dot{B}_t) d \left( \frac{H^x_t}{G^x(t)} \right) + H^x_t \frac{G^x(T)}{G^x(t)} \frac{\partial \dot{Q}(t, T, \dot{B}_t)}{\partial B_t} d\dot{B}_t + G^x(T) \dot{Q}(t, T, \dot{B}_t) d \left( \frac{H^x_t}{G^x(t)} \right).
\]

It follows from equation (3.10) that the probability density of \( \xi_x \) on the interval \([s, \infty)\), conditional on \( A_s \), equals \( \frac{G^x(u)}{G^x(s)} \) for \( u \in [s, \infty) \). Consequently, the drift of the stochastic differential equation (SDE) for \( \frac{H^x_t}{G^x(t)} \) given by

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} E \left( \frac{H^x_{s+\Delta}}{G^x(s+\Delta)} - \frac{H^x_s}{G^x(s)} \right| A_s)
\]

\[
= \lim_{\Delta \to 0} \frac{1}{\Delta} E \left( - \int_{s}^{s+\Delta} H^x_u \frac{(G^x(u))'}{(G^x(u))^2} du + \int_{s}^{s+\Delta} \frac{1}{G^x(u)} dH^x_u \right| A_s) = 0
\]

vanishes. Therefore, the process \( M^2 = \{ M^2_t, t \geq 0 \} \), where

\[
M^2_t = G^x(T) \int_0^T \dot{Q}(u, T, \dot{B}_u) d \left( \frac{H^x_u}{G^x(u)} \right),
\]

is a local martingale. Consequently, one has \( M^1 = \{ M^1_t, t \geq 0 \} \) as the difference of two local martingales, forming a local martingale. This is completing the proof because a product of \( M^2 \) with any benchmarked traded self-financing portfolio turns out to be driftless and, thus, to be a local martingale. \( \Box \)
From the martingale representation in Proposition 3.1 it follows as in Du & Platen (2012) that the trading strategy \( \delta = \{ \delta_t = (\delta^0_t, \delta^1_t), t \in [0, T] \} \), where

\[
\delta^0_t = \frac{G^x(T)}{G^x(t)} H^x_t \frac{\partial Q(t, \hat{B}_t)}{\partial \hat{B}_t}
\]

and

\[
\delta^1_t = \hat{V}_t - \hat{C}_t - \delta^0_t \hat{B}_t,
\]

is the investment strategy that results in the minimal initial cost for hedging the claim. Furthermore, the benchmarked P&L process equals \( \hat{C}_t = M^2_t \). It is minimal in various ways, for instance, as measured by the derivative of the predictable projection, see Revuz & Yor (1999), of the quadratic variation of the benchmarked P&L.

More intuitively, consider the square of the benchmarked P&L when substituting (3.14) into (3.13), yielding

\[
(\hat{C}_t)^2 = \left( E \left( H^x_T \hat{B}_T \right) - \hat{V}_0 \right)^2 + 2 \left( E \left( H^x_T \hat{B}_T \right) - \hat{V}_0 \right) \left( \int_0^t \left( \frac{\partial \hat{V}(u, T, \hat{B}_u)}{\partial \hat{B}_u} - \delta^0_u \right) d\hat{B}_u + \int_0^t \hat{V}_u d \left( \frac{H^x_u}{G^x(u)} \right) \right) \\
+ \left( \int_0^t \left( \frac{\partial \hat{V}(u, T, \hat{B}_u)}{\partial \hat{B}_u} - \delta^0_u \right) d\hat{B}_u \right)^2 \\
+ 2 \int_0^t \left( \frac{\partial \hat{V}(u, T, \hat{B}_u)}{\partial \hat{B}_u} - \delta^0_u \right) d\hat{B}_u \int_0^t \hat{V}_u d \left( \frac{H^x_u}{G^x(u)} \right) + \left( \int_0^t \hat{V}_u d \left( \frac{H^x_u}{G^x(u)} \right) \right)^2.
\]

If one could take expectations on both sides of this equation, one would realize that BRM would minimize the second moment of the benchmarked P&L. Obviously, the second and fourth term would vanish. The first term would provide the price. The third term would yield the hedge ratio \( \delta^0 \). The fifth term could not be further reduced since \( \hat{V} \) would represent the minimal possible benchmarked price. Obviously, this minimization would identify the benchmarked P&L as \( M^2_t \). The concept of orthogonality, as applied in BRM, provides the same outcome under considerably weaker technical assumptions.

For a large pension scheme, the aggregate benchmarked portfolio of the scheme is typically comprised of many benchmarked P&L accounts. The policy holders’ lifetimes can be considered to be independent. For the benchmarked portfolio with independent mortality risks, the resulting benchmarked aggregate P&L process asymptotically vanishes relative to the entire benchmarked portfolio value as the number of members in the scheme increases. This diversification effect is a result of pooling and in line with the classical insurance argument and investigated in Section 5. Furthermore, from the beginning of post retirement payments until the death of the member the pension scheme member makes usually many,
say monthly, payments, which creates some smoothing effect for the assets of the scheme.

In summary, BRM allows us to identify, for a given liability, the investment strategy which results in the minimal price and the, in a practical sense, minimal benchmarked nonhedgeable part. If the mortality were modeled correctly and the number of members were infinite, the mortality risk could be entirely removed. In reality, there exist such and other uncertainties that cannot be removed, as we will discuss in the next section.

4 Least Expensive Pensions

In the way as indicated in the previous, one could price and hedge post retirement payments for all members of a pension scheme, e.g. assuming a life annuity for each member of the scheme paying a respective number of units of the savings account each month post retirement until death. However, in reality, there are additional practical considerations and uncertainties that have to be taken into account when designing a sustainable, least expensive, pension scheme: As reality changes, model parameters change and need to be updated, e.g. due to increasing longevity. Models can become very accurate but will never be perfect. Hedging cannot be performed continuously as assumed in Sections 2 and 3, and hedge errors due to discrete time hedging arise. Trading orders take time to be processed. These are only a few practical considerations, and other uncertainties may arise, which a least expensive pension scheme will have to absorb.

Consequently, for many reasons deviations from the theoretically modeled pension fund evolution will arise. From the point of view of the members of a pension scheme, the sustainability of the scheme, i.e. its ability to service its liabilities, is paramount. As a result, the "errors" have to be absorbed by some entity. We argue in this paper that this should be the pension scheme itself, and we introduce the concept of least expensive pensions. The idea is that payouts are not guaranteed, they are only targeted and entitlement levels can be regularly calculated and adjusted by matching assets with liabilities. The members of the scheme jointly share then the risks which emerge over time and cannot be hedged. This idea is not completely new. However, what is new is that the modeling and hedging of a least expensive pension can be made in a precise manner fair and transparent. By using the modeling freedom of the benchmark approach, post retirement payments can be significantly higher than provided by fully classical schemes. Least expensive schemes will be shown to be most natural. The aim of this paper is to point out what potential extra benefits can be expected when valuing and hedging liabilities under the proposed approach.

To illustrate clearly the design and properties of the type of least expensive pension schemes that we propose, we do not spread post retirement payments over
time. In reality one has usually far more smoothing going on in a pension scheme. Assume that at time 0, the pension scheme has \( n \) members, where each member is entitled to receive one unit of the savings account at the fixed time \( T > 0 \). For a least expensive pension, payout levels are not guaranteed, they are only targeted. Crucial is the calculation of the, so called, entitlement level \( \beta_t \), at times \( t \in [0, T] \). By matching assets with liabilities at time \( t \) it determines how many units of the savings account are at time \( t \) expected to be paid to each member at the payout date \( T \). This entitlement level will fluctuate over time, absorbing hedge errors, model changes, changes in parameter values, changes in government regulation, etc. In reality, the entitlement level can only be adjusted at sufficiently frequent discrete times, \( t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots \). Between adjustment dates, the level will be kept constant at the value determined at the most recent adjustment date.

As a result, the benchmarked total liability of the pension scheme at time \( t < T \), comprised of \( n \) members, where the \( i \)th member is aged \( x_i \) at time 0, all entitled to one unit of the savings account at time \( T \), is given by

\[
\beta_t \sum_{i=1}^{n} \frac{G(x_i(T))}{G(x_i(t))} H(x_i(t)) \tilde{B}_t E \left( \tilde{B}_T | \mathcal{A}_t \right).
\]

The total liability is hedged using an investment strategy composed by those strategies outlined in Section 3. At time \( t_k \), the pension scheme invests \( \tilde{\delta}^0_{t_k} \) units in the savings account and \( \tilde{\delta}^1_{t_k} \) units in the GOP, resulting in a benchmarked total value \( \hat{R}_{t_k} \) of the scheme’s assets which satisfies the relation

\[
\hat{R}_{t_k} = \tilde{\delta}^0_{t_k} \tilde{B}_{t_k} + \tilde{\delta}^1_{t_k}.
\]

At time \( t_k \), the entitlement level \( \beta_{t_k} \) is obtained by the requirement that assets have to match liabilities, which leads to the formula

\[
\beta_{t_k} = \frac{\tilde{\delta}^0_{t_k} \tilde{B}_{t_k} + \tilde{\delta}^1_{t_k}}{\sum_{i=1}^{n} \frac{G(x_i(T))}{G(x_i(t_k))} H(x_i(t_k)) \tilde{B}_t E \left( \tilde{B}_T | \mathcal{A}_{t_k} \right)}.
\] (4.16)

During the time interval \([t_k, t_{k+1})\), the assets are invested according to the strategy set up at time \( t_k \). At time \( t_{k+1} \), the assets are rebalanced and a new entitlement level \( \beta_{t_{k+1}} \) is calculated according to equation (4.16). For the scheme members, it is eminent that the price they pay for their benefits is minimal and the entitlement level does not fluctuate much over time. The previously discussed concept of benchmarked risk minimization (BRM) achieves these goals, as will be illustrated in the next section.

The proposed approach identifies the minimal cost of the liabilities of the scheme at any given time, which can be significantly below those obtained using the classical risk neutral approach. Also this property will be demonstrated in the next section. Furthermore, by regularly adjusting the entitlement level \( \beta_t \), the
members of the pension scheme share the risks and benefits due to unhedgeable uncertainties in a fair and transparent manner.

It is clear that the entitlement level at a given time \( t \in [0, T] \) allows also new entrants to the scheme to join the scheme in a fair and transparent manner. Similarly, if a member wishes to exit the scheme, the entitlement level describes in a fair manner what amount can be withdrawn without distorting the overall balance. This makes a least expensive pension highly portable. However, the main point of this paper is that the post retirement payouts of a pension scheme can be made larger than those that traditional schemes aim for. At this point, it seems to be appropriate to discuss current funding levels of pension schemes. As mentioned in Working Party (2007), many pension schemes have substantial deficits, as measured by IAS 19/FRS17 on a bond related basis. It is expected that this situation will further deteriorate overtime if no change will be made. As will be indicated in the next section, the situation would improve significantly if key ideas of this paper would be implemented.

5 Case Study: Australia

The aim of this section is to indicate how the approach presented in this paper could influence positively the long term performance of pension schemes. Using Australian financial and mortality data, we show how a model would work for the above given example. We emphasize that due to the tractability of the model under consideration and the simplicity of the structure of the pension scheme, the procedure is very transparent and can be easily performed. Subsequently, we investigate the investment strategies proposed. In particular, we confirm that the benchmarked aggregate P&L process vanishes approximately. Finally, we show by using historical data that for a diversified pension scheme, the entitlement level experiences little fluctuations.

5.1 Fitting the Model

In this subsection, we fit the minimal market model (MMM), as discussed in Section 2, and the threshold life table model, as discussed in Section 3. Regarding the threshold life table model, we employ mortality data for Australia from the Human Mortality Database (2011). We fit the model using weighted least squares estimation, as described in Li, Hardy & Tan (2008), and obtain the parameters,

\[
N = 91, \ b = 4.26 \times 10^{-6}, \ c = 1.123117, \ \gamma = -0.36412, \ \vartheta = 5.49884.
\]

Figure 5.3 compares the fitted survival probabilities \( \frac{G^x(t)}{G^x(T)} \), where \( x = 40 \), for \( t = 0, \ldots, 60 \) and \( T = 60 \) to the data. We remark that the probabilities \( \frac{G^x(t)}{G^x(T)} \),
where $x = 40$ and $t = 0, \ldots, 37$ and $T = 37$, a subset of the probabilities shown in Figure 5.3, are the ones we use in Subsection 5.2. We note that the model fits visually the data well.

Regarding the financial data, we employ as proxy for the GOP the equi-weighted index (EWI), constructed in Le & Platen (2006) and Platen & Rendek (2008), which uses 114 world industry sector indices as constituents provided by Datasync Thomson Financial for the period from 01/01/1973 to 18/03/2010. We remark that 40 basis points proportional transaction costs per trade are applied in the construction of the EWI. The EWI is converted into Australian dollars using FX data from Datastream Thomson Financial and Australian interest rate data as obtained from the same source. We estimate the parameters of the MMM, $\alpha_0$ and $\eta$, as follows, see also Platen & Heath (2010): the quadratic variation of the square-root of the discounted GOP $\left[\sqrt{S}\right]_t$ satisfies

$$\varphi(t) - \varphi(0) = \left[\sqrt{S}\right]_t = \frac{1}{4} \int_0^t \alpha_s ds = \frac{\alpha_0}{4\eta} \left(\exp \{\eta t\} - 1\right), \quad (5.17)$$

see Platen & Heath (2010). Therefore, equation (5.17) allows us to estimate $\alpha_0$ and $\eta$ from a least squares fit. We obtain $\alpha_0 = 0.7786$, $\eta = 0.1056$. For simplicity, we set $\varphi(0) = 0$. In Figure 5.4, we plot the quadratic variation of the square-root of the discounted GOP versus the fitted $\varphi(t)$, and find that our parameters provide a good fit. Having fitted the mortality model and the financial model, we are now in a position to assess the model.
5.2 Hedging the Payoff

In this subsection, we discuss how to hedge a mortality-linked savings account payoffs. The liability is hedged discretely, at time points \( t_k = \frac{k}{260}, \ k = 0, \ldots, 9708 \), i.e. we rebalance the portfolio on a daily basis for the period \( t_0 = \text{01/01/1973} \) until \( T = \text{18/03/2010} \). We denote the benchmarked hedge portfolio, defined on the discrete time grid \( \{t_k, k = 0, \ldots, 9708\} \), by \( \tilde{Q}_{t_k} \). Here we set

\[
\tilde{Q}_{t_0} = \frac{G^x(T)}{G^x(t_0)} \hat{Q}_{t_0},
\]

\[
\tilde{Q}_{t_k} = \tilde{Q}_{t_{k-1}} \left( 1 + \frac{G^x(T) \delta_t^0 \hat{B}_{t_{k-1}}}{G^x(t_{k-1}) \hat{Q}_{t_{k-1}} \left( \frac{\hat{B}_{t_k}}{B_{t_{k-1}}} - 1 \right)} \right),
\]

for \( k = 1, \ldots, 9708 \), where

\[
\delta_t^0 = \frac{\partial \tilde{Q}(t, T, \hat{B}_t)}{\partial B_t},
\]

see Section 3.

Now, we consider the potential performance of pension schemes with respect to diversification over age. We assume that the scheme is comprised of the deliberately low number of 4500 members at time \( t_0 \) with 100 members aged 20, 100 members aged 21, 100 members aged 22, \ldots, 100 members aged 63, and 100 members aged 64. We assume that time \( t_0 \) corresponds to \( \text{01/01/1973} \) and that all members are entitled to one unit of the savings account falling due on \( \text{18/03/2010} \). Of course, in a more realistic situation one would have streams of pension payouts, distributed over time and not lump sums as in our case. This would give much smoother results than the extreme case of lump sum payouts.
only at one date that we consider here. In Figure 5.5, we show the benchmarked aggregate hedge portfolio and also the benchmarked aggregate P&L process: We find that the pooling of pension contracts makes assets matching liabilities rather well. Hence, we only plot the hedging portfolio, as the value of the liability would be visually indistinguishable from the former. We note in the right panel of Figure 5.5 that the benchmarked P&L process approximately vanishes. From the point of view of a pension scheme, this is very desirable. The nonhedgeable mortality risk, aggregated in the benchmarked P&L process, vanishes approximately through pooling for a sufficiently large number of members, as under the classical approach.

Finally, we examine the entitlement level of a respective sustainable pension scheme. We recall from our previous discussions and Figure 5.5, that even for a reasonably large number of members, there is some remaining risk that assets may not match targeted liabilities. By transferring the remaining mortality risk and potential other risks to the pension scheme members through adjusting the entitlement level regularly, these risks but also potential gains are shared.

We recall that the entitlement level $\beta_t$ ensures that assets meet liabilities, see equation (4.16). It is important to the members, that $\beta_t$ fluctuates only little over time. When implementing the above discussed pension scheme and simulating the deaths according to the mortality table. Figure 5.6 shows the resulting entitlement level $\beta_t$. We observe that $\beta_t$ fluctuates little, the largest deviation from the initial level one over the 37 year period is less than 3.5%. Furthermore, we observe that at sudden market downturns, e.g. at the 1987 market crash, the hedge portfolio reacts with an upward move in the entitlement level. This is a consequence of the dynamics of the underlying model and the fact that we are not modeling jumps in this paper. The fluctuations of the entitlement level shown in Figure 5.6 illustrate that the remaining mortality risk and the hedge errors from discrete hedging impact the entitlement level only very little. Of course, in real-
ity, other risks will create additional changes in the entitlement level. However, almost any pension scheme will have to face these other risks. Least expensive pensions will maximize the payout and will share the other risks in a fair and transparent manner. In summary, one can conclude that pricing and hedging of least expensive pensions with less expensively generated long term payouts is a realistic alternative to traditional pensions. It needs to be emphasized that the mutual nature of the proposed type of schemes makes these less expensive than any other scheme, where external entities have to balance potential shortfalls in the scheme and classical pricing is employed. With the example of least expensive pensions one can see how liability driven investment can be more efficiently managed than currently practised. There is strong potential that pension schemes that currently have difficulties in matching with their assets their liabilities under classical modeling, may be able to achieve the overall goal of the pension by going into the direction proposed in this paper.

References


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